

Exercise 6: Statistical inference (III)

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Part 1: Wald, Score, and likelihood ratio test statistics

Write out the likelihood function, and derive the test statistics of the Wald, Score, and likelihood ratio test.

1. $X_i \stackrel{\text{i.i.d.}}{\sim} f(x | \theta)$

$$f(x | \theta) = \theta \exp(-x\theta) \mathbb{I}\{x > 0\}$$

2. $X_i \stackrel{\text{i.i.d.}}{\sim} f(x | \theta)$

$$f(x | \theta) = \theta c^\theta x^{-(\theta+1)} \mathbb{I}\{x > c\} \quad (\text{Pareto distribution})$$

where c is a known constant and θ is unknown.

Part 2: Test equivalence

Let θ be a scalar parameter and suppose we test

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta \neq \theta_0.$$

Let W be the Wald test statistic and let λ be the likelihood ratio test statistic. Show that these tests are equivalent in the sense that

$$\frac{W^2}{\lambda} \xrightarrow{\text{P}} 1$$

as $n \rightarrow \infty$. Hint: Use a Taylor expansion of the log-likelihood $\ell(\theta)$ to show that

$$\lambda \approx \left(\sqrt{n} (\hat{\theta} - \theta_0) \right)^2 \left(-\frac{1}{n} \ell''(\hat{\theta}) \right)$$