

## Exercise 6: Statistical inference (III)

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### Part 1: Wald, Score, and likelihood ratio test statistics

Write out the likelihood function, and derive the test statistics of the Wald, Score, and likelihood ratio test.

1.  $X_i \stackrel{\text{i.i.d.}}{\sim} f(x \mid \theta)$

$$f(x \mid \theta) = \theta \exp(-x\theta) \mathbb{I}\{x > 0\}$$

2.  $X_i \stackrel{\text{i.i.d.}}{\sim} f(x \mid \theta)$

$$f(x \mid \theta) = \theta c^\theta x^{-(\theta+1)} \mathbb{I}\{x > c\} \quad (\text{Pareto distribution})$$

where  $c$  is a known constant and  $\theta$  is unknown.

### Part 2: Test equivalence

Let  $\theta$  be a scalar parameter and suppose we test

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta \neq \theta_0.$$

Let  $W$  be the Wald test statistic and let  $\lambda$  be the likelihood ratio test statistic. Show that these tests are equivalent in the sense that

$$\frac{W^2}{\lambda} \xrightarrow{\text{P}} 1$$

as  $n \rightarrow \infty$ . Hint: Use a Taylor expansion of the log-likelihood  $\ell(\theta)$  to show that

$$\lambda \approx \left( \sqrt{n} (\hat{\theta} - \theta_0) \right)^2 \left( -\frac{1}{n} \ell''(\hat{\theta}) \right)$$