Solution 9: Parallel Computing and Simulations

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Simulations on Cauchy

Suppose $\boldsymbol{X} = (X_1, \ldots, X_n)$ is an i.i.d. sample from the shifted Cauchy distribution with density

$$f(x \mid \theta) = \frac{1}{\pi\left(1 + (x - \theta)^2\right)}, \quad x \in \mathbb{R}$$

Our goal is to compare the following 4 estimators of the parameter θ.

- Sample mean

$$\hat{\theta}_n^{(1)} = \bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

- Sample median

$$\hat{\theta}_{\mathrm{n}}^{(2)} = M_{\mathrm{n}} = \frac{1}{2}\left(X_{(k)} + X_{(k+1)}\right)$$

- Modified sample mean

$$\hat{\theta}_n^{(3)} = M_n + \frac{2}{n} \cdot \left.\frac{\partial \ell}{\partial \theta}\right|_{\theta = M_n}$$

  where ℓ is the log-likelihood function.

- Maximum likelihood estimator (MLE) $\hat{\theta}_{\mathrm{n}}^{(4)}$ defined by

$$\ell\left(\hat{\theta}_n^{(4)} \mid X\right) = \max_{\theta \in \mathbb{R}} \ell(\theta \mid \boldsymbol{X})$$

  where ℓ is the log-likelihood function.

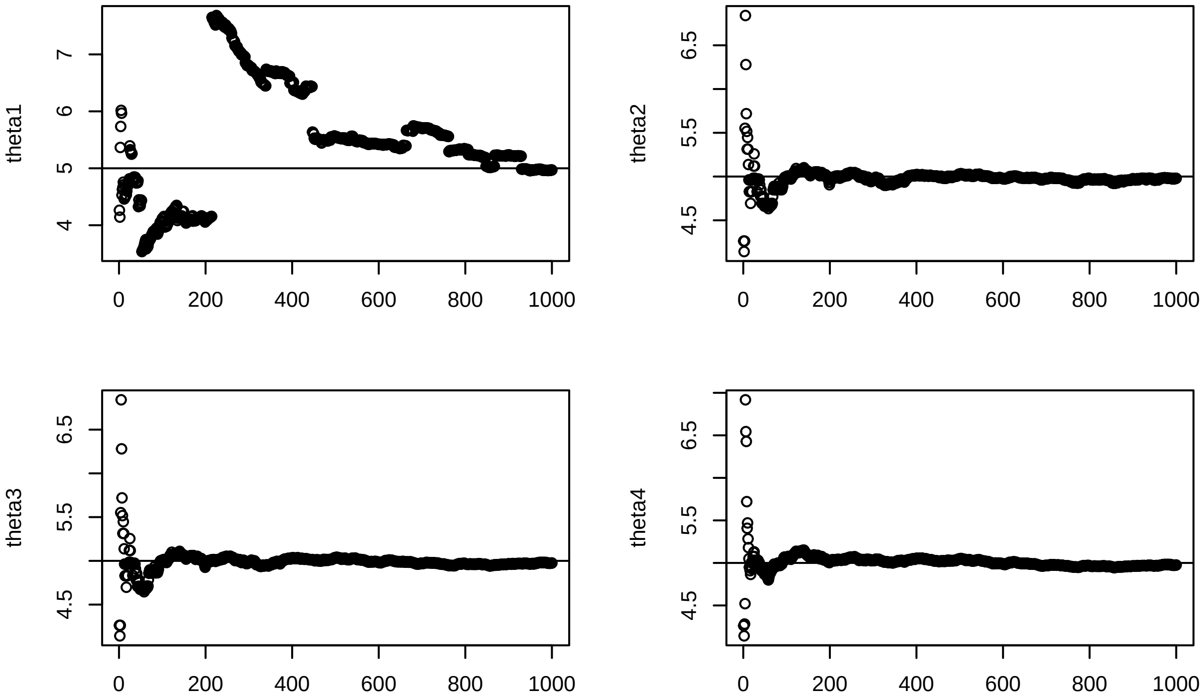
1. Derive the likelihood function and log-likelihood function.
2. Simulate data from Cauchy distribution with location 5, and scale 1.
3. Choose your number of simulations.
4. Verify consistency of the estimators.
5. Calculate the mean square error of the estimators.
6. Calculate the coverage probability of the estimators. Calculate $\mathbf{P}_\theta\left(\left|\hat{\theta}_n - \theta\right| \leq \varepsilon\right)$, for $\varepsilon = 0.1$, and $\theta = 5$.

Solution

```r
# Simulate Cauchy.
n <- 1000
set.seed(123456)
cauchy_samples <- rcauchy(n, location = 5, scale = 1)

# Function to calculate the derivative of the log-likelihood.
```

```r
dloglik <- function(x, theta) {
  d1 <- x - theta
  d2 <- 1 + (x - theta)^2
  return(2*sum(d1/d2))
}

# Function to calculate the log-likelihood.
loglik <- function(theta) {
   x <- cauchy_samples
   n <- length(x)
   l1 <- -n*log(pi)
   l2 <- -sum(log(1 + (x - theta)^2))
   return(l1 + l2)
}

nloglik<- function(theta) {
  return(-loglik(theta))
}
```

```r
# Consistency.
theta1 <- vector(length = n)
theta2 <- vector(length = n)
theta3 <- vector(length = n)
theta4 <- vector(length = n)

full_samples <- cauchy_samples
for (k in 1:n) {
  cauchy_samples <- NULL
  cauchy_samples <- full_samples[1:k]
  #print(cauchy_samples)
  theta1[k] <- mean(cauchy_samples)
  #print(theta1[i])
  theta2[k] <- median(cauchy_samples)
  theta3[k] <- theta2[k] + 2/n * dloglik(cauchy_samples, theta2[k])
  theta4[k] <- optimize(nloglik, interval = c(-10,10))$minimum
}

par(mfrow = c(4, 2), mar = c(1, 4.1, 4.1, 2.1))
plot(theta1)
abline(h = 5)
plot(theta2)
abline(h = 5)
plot(theta3)
abline(h = 5)
plot(theta4)
abline(h = 5)
```



```r
nsim <- 1000
theta1 <- vector(length = nsim)
theta2 <- vector(length = nsim)
theta3 <- vector(length = nsim)
theta4 <- vector(length = nsim)

for (i in 1:nsim) {
  set.seed(1234 + i)

  n <- 1000
  cauchy_samples <- rcauchy(n, location = 5, scale = 1)

  theta1[i] <- mean(cauchy_samples)
  theta2[i] <- median(cauchy_samples)
  theta3[i] <- theta2[i] + 2/n * dloglik(cauchy_samples, theta2[i])
  theta4[i] <- optimize(nloglik, interval = c(-10,10))$minimum
}
```

```r
mse <- function(hat_theta, theta) {
  return(mean(hat_theta - theta)^2)
}

cat("mse1 =", mse(theta1, 5), "\n")
```

```
## mse1 = 1.530928
```

```r
cat("mse2 =", mse(theta2, 5), "\n")
```

```
## mse2 = 6.172925e-06
```

```r
cat("mse3 =", mse(theta3, 5), "\n")
```

```
## mse3 = 1.968817e-06
```

```r
cat("mse4 =", mse(theta4, 5), "\n")
```

```
## mse4 = 1.835778e-06
```

```r
# Coverage probability
cov_prob <- function(hat_theta, theta, eps = 0.1) {
  return(mean(abs(hat_theta - theta) <= eps))
}

cat("cov_prob1 =", cov_prob(theta1, 5), "\n")
```

```
## cov_prob1 = 0.064
```

```r
cat("cov_prob2 =", cov_prob(theta2, 5), "\n")
```

```
## cov_prob2 = 0.957
```

```r
cat("cov_prob3 =", cov_prob(theta3, 5), "\n")
```

```
## cov_prob3 = 0.972
```

```r
cat("cov_prob4 =", cov_prob(theta4, 5), "\n")
```

```
## cov_prob4 = 0.973
```