Supplement 3: Wald, Score and LR Test Equivalence

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The Wald test

We are interested in testing the hypotheses in a parametric model:

$$H_0 : \theta = \theta_0 \quad \text{versus } H_1 : \theta \neq \theta_0.$$

The Wald test most generally is based on an asymptotically normal estimator, i.e. we suppose that we have access to an estimator $\widehat{\theta}$ which under the null satisfies the property that:

$$\widehat{\theta} \xrightarrow{d} N\left(\theta_0, \sigma_0^2\right)$$

where σ_0^2 is the variance of the estimator under the null. The canonical example is when $\widehat{\theta}$ is taken to be the MLE.

The Wald statistic

If the hypothesis involves only a single parameter restriction, then the Wald statistic takes the following form:

$$W_n = \frac{\left(\hat{\theta} - \theta_0\right)^2}{\mathrm{var}(\hat{\theta})},$$

which under the null hypothesis follows an asymptotic χ_1^2-distribution with one degree of freedom.

Example

Suppose we considered the problem of testing the parameter of a Bernoulli, i.e. we observe $X_1, \ldots, X_n \sim \text{Ber}(p)$, and the null is that $p = p_0$. Defining $\widehat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i$. A Wald test could be constructed based on the statistic:

$$T_n = \frac{(\widehat{p} - p_0)^2}{\frac{p_0(1-p_0)}{n}},$$

which has an asymptotic χ_1^2 distribution. An alternative would be to use a slightly different estimated standard deviation, i.e. to define,

$$T_n = \frac{(\widehat{p} - p_0)^2}{\frac{\widehat{p}\left(1-\widehat{p}\right)}{n}},$$

Observe that this alternative test statistic also has an asymptotically χ_1^2 distribution under the null.

The score test

Score test is based on the value of the score function $U(\theta)$ under the null hypothesis H_0.

Reminder: $U(\theta) = \ell'(\theta)$.

The score test statistic

$$S_n = \frac{U(\theta_0)^2}{\text{var}[U(\theta_0)]},$$

which has an asymptotic distribution of χ_1^2 under the null.

Reminder: the variance of the score function is the Fisher information.

The score test

Similarly, we have

$$\widehat{I}\left(\theta_0\right) = -\frac{1}{n}\sum_{i=1}^{n} \left.\frac{\partial^2 \log f\left(x_i \mid \theta\right)}{\partial\theta^2}\right|_{\theta_0} \xrightarrow{\mathrm{P}} I\left(\theta_0\right)$$

so that

$$S_n = \frac{U(\theta_0)^2}{\widehat{I}\left(\theta_0\right)}$$

also has an asymptotic distribution of χ_1^2 under the null.

The likelihood ratio test

Let $\hat{\theta}_n$ be the MLE of θ.

$$\Delta_n = \ell\left(\widehat{\theta}_n\right) - \ell\left(\theta_0\right) = \log\left(\frac{L(\hat{\theta}_n \mid \mathbf{x})}{L\left(\theta_0 \mid \mathbf{x}\right)}\right) = \log\left(\frac{\sup_{\theta\in\Theta} L(\theta \mid \mathbf{x})}{L\left(\theta_0 \mid \mathbf{x}\right)}\right) \geq 0$$

Under H_0,

$$2\Delta_n \xrightarrow{\mathrm{D}} \chi^2_1$$

The Wald test, score test, and likelihood ratio test

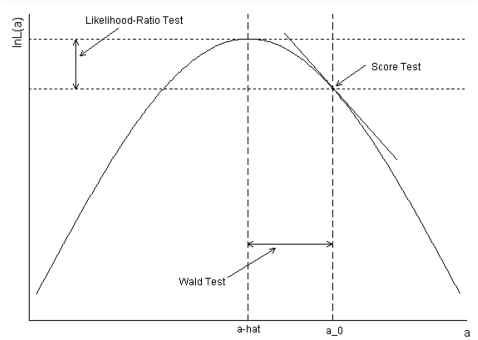

Figure 1: Fox, J. (1997) Applied regression analysis, linear models, and related methods. Thousand Oaks, CA: Sage Publications. P. 570.

- Wald Test: Horizontal distance between $\hat{a}$ and a_0.
- Likelihood-Ratio Test: Vertical drop from $\hat{a}$ to a_0.
- Score Test: Slope at a_0.

Test equivalence

We can show that (when there is no misspecification)

- The tests are asymptotically equivalent in the sense that under H_0 they reach the same decision with probability 1 as $n \to \infty$.
- For a finite sample size n, they have some relative advantages and disadvantages with respect to one another.

Discussion

$$W_n = \frac{\left(\hat{\theta} - \theta_0\right)^2}{\text{var}(\hat{\theta})} \xrightarrow{\text{D}} \chi_1^2$$

$$S_n = \frac{U(\theta_0)^2}{\widehat{I}\left(\theta_0\right)} \xrightarrow{\text{D}} \chi_1^2$$

$$2\Delta_n = 2\left\{\ell\left(\widehat{\theta}_n\right) - \ell\left(\theta_0\right)\right\} \xrightarrow{\text{D}} \chi_1^2$$

- It is easy to create one-sided Wald and score tests.
- The score test does not require $\widehat{\theta}_n$ whereas the other two tests do.
- The Wald test is most easily interpretable and yields immediate confidence intervals.
- The score test and LR test are invariant under reparametrization, whereas the Wald test is not.

Resources

This tutorial is based on

- "All of statistics" Chapter 10 by Larry A. Wasserman.
- Arnaud Doucet's STA 461 Lecture notes [links].