Exercises for Module 1: Proofs

1. Prove De Morgan's Laws for propositions: $\neg(P \wedge Q) = \neg P \vee \neg Q$ and $\neg(P \vee Q) = \neg P \wedge \neg Q$ (Hint: use truth tables).

2. Write the following statements and their negations using logical quantifier notation and then prove or disprove them:
(i) Every odd integer is divisible by three.

(ii) For any real number, twice its square plus twice itself plus six is greater than or equal to five. *(You may assume knowledge of calculus.)*

(iii) Every integer can be written as a unique difference of two natural numbers.

3. Prove the following statements:
(i) If $a|b$ and $a, b \in \mathbb{N}$ (positive integers), then $a \leq b$.

(ii) If $a|b$ and $a|c$, then $a|(xb+yc)$, where $a, b, c, x, y \in \mathbb{Z}$.

(iii) Let $a, b, n \in \mathbb{Z}$. If n does not divide the product ab, then n does not divide a and n does not divide b.

4. Prove that for all integers $n \geq 1$, $3|(2^{2n}-1)$.

5. Prove the Fundamental Theorem of Arithmetic, that every integer $n \geq 2$ has a unique prime factorization (i.e. prove that the prime factorization from the last proof is unique).

6. Let $A = \{x \in \mathbb{R} : x < 100\}$, $B = \{x \in \mathbb{Z} : |x| \geq 20\}$, and $C = \{y \in \mathbb{N} : y \text{ is prime}\}$ $(A, B, C \subseteq \mathbb{R})$. Find $A \cap B$, $B^c \cap C$, $B \cup C$, and $(A \cup B)^c$.