Module 10: Differentiation and Integration

Operational math bootcamp

Statistical Sciences
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Outline

- Differentiation on $\mathbb{R}$
  - Mean value theorem
  - l'Hôpital's rule
  - Smoothness classes
- Integration on $\mathbb{R}$
  - Riemann sums and Riemann integral
  - Integration rules
  - Drawbacks of Riemann integration

Differentiation

Derivative

Recall the definition of the derivative:

Definition

A function $f : (a, b) \to \mathbb{R}$ is *differentiable* at $x \in (a, b)$ if

$$L := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. L is the *derivative* of f at x, denoted $L = f'(x)$. If f is differentiable at every $x \in (a, b)$, we say f is *differentiable*.

Proposition

The following are key rules for differentiation:

1. If f is differentiable at x, then it is continuous at x.
2. The derivative of a constant function is zero.
3. If f and g are differentiable at x, then so is $f+g$ with $(f+g)'(x) = f'(x) + g'(x)$.
4. Product rule: If f and g are differentiable at x, then so is fg with
$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$
5. Quotient rule: If f and g are differentiable at x and $g(x) \neq 0$, then so is f/g with
$$(f/g)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$
6. Chain rule: If f is differentiable at x and g is differentiable at $f(x)$, then so is $g \circ f$ with
$$(g \circ f)'(x) = g'(f(x))f'(x).$$

Lemma

If $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b) and achieves a (local) maximum or (local) minimum at $c \in (a, b)$, then $f'(c) = 0$.

Proof

Proof continued

Next we recall a theorem we proved in greater generality when we discussed compactness in the topology section:

Theorem (Extreme value theorem)

If $f : [a, b] \to \mathbb{R}$ is continuous, then f attains a maximum and a minimum, i.e. there exists $c, d \in [a, b]$ such that

$$f(c) \leq f(x) \leq f(d) \quad \forall x \in [a, b].$$

This theorem is used to prove the following important result:

Theorem (Mean value theorem)

If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

Proof.

$\square$

Corollary

If $f : (a, b) \to \mathbb{R}$ is differentiable and has a bounded derivative (i.e. $|f'(x)| \leq M$ for some $M > 0$ and for all $x \in (a, b)$), then f is Lipschitz.

Proof.

$\square$

l'Hôpital's rule

Theorem (l'Hôpital's rule)

If f, g are differentiable on (a, b), where a, b may be $\pm\infty$, and $\lim_{x\to b} f(x) = 0 = \lim_{x\to b} g(x)$, or both limits equal $\pm\infty$, then

$$\lim_{x\to b} \frac{f'(x)}{g'(x)} = L$$

implies

$$\lim_{x\to b} \frac{f(x)}{g(x)} = L$$

Example

$$\lim_{x \to 0} \frac{5^x - 2^x}{x^2 - x}$$

$$\lim_{x \to -\infty} xe^x$$

Higher order derivatives

Definition

We define higher-order derivatives inductively as $f^{(r)}(x) = (f^{(r-1)})'(x)$. If $f^{(r)}$ exists (at x), we say that f is r^{th}-order differentiable (at x).

Definition

If $f^{(r)}$ exists for all $r \in \mathbb{N}$ and for all $x \in (a, b)$, then we say f is infinitely differentiable or *smooth*. We denote this $f \in C^{\infty}$.

Smoothness classes

Definition

If f is differentiable and its derivative $f'(x)$ is continuous, we say that f is *continuously differentiable*, and that $f \in C^1$. If $f^{(r)}$ exists and is continuous, we say that $f \in C^r$. If f is continuous, we say $f \in C^0$.

Since differentiability implies continuity, we have $C^\infty \subset \cdots \subset C^2 \subset C^1 \subset C^0$.

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Example

- The function $f(x) = |x|$ is C^0 but not C^1.
- The function $f(x) = x|x|$ is C^1 but not C^2.
- $f(x) = e^x$ and $f(x) = x$ are smooth functions, i.e. in C^∞.

Integration

Riemann integration

Definition (Riemann sum)

Let $f : [a, b] \to \mathbb{R}$ be a function. We call a set of points $P = \{x_0, \ldots, x_n\} \subseteq [a, b]$ a *partition* of $[a, b]$ if the following holds

$$a = x_0 \leq x_1 \leq \ldots \leq x_{n-1} \leq x_n = b.$$

We call the largest sub-interval of the partition P the *mesh* of P, denoted $|P|$, i.e.

$$|P| = \max_{i=1,\ldots,n} (x_i - x_{i-1}).$$

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Definition continued (Riemann sum)

Given a partition $P = \{x_0, \ldots, x_n\} \subseteq [a, b]$ of $[a, b]$ and a set of points $T = \{t_1, \ldots, t_n\} \subseteq [a, b]$ such that $t_i \in [x_{i-1}, x_i]$ for $i = 1, \ldots, n$, we define the *Riemann sum* $R(f, P, T)$ corresponding to f, P, T as

$$R(f, P, T) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) := \sum_{i=1}^{n} f(t_i)\Delta x_i,$$

where we used $\Delta x_i = x_i - x_{i-1}$.

The idea is to define the Riemann integral as the "limit" of the Riemann sums over all partition such that their mesh is becoming arbitrarily small:

Definition (Riemann integrable)

A function $f\colon [a,b] \to \mathbb{R}$ is called *Riemann integrable* if there exists $I \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for any partition $P = \{x_0, \dots, x_n\}$ of $[a,b]$ with $|P| < \delta$ and set of points $T = \{t_1, \dots, t_n\} \subseteq [a,b]$ such that $t_i \in [x_{i-1}, x_i]$ for $i = 1, \dots, n$ we have $|R(f,P,T) - I| < \epsilon$.
We say that I is the Riemann integral of f, denoted $I = \int_a^b f(x)\mathrm{d}x$.

If f is Riemann integrable, then I is unique.

Let $\mathcal{R}([a,b])$ denote the set of functions that are Riemann integrable on $[a,b]$.

Theorem

Riemann integration is linear, i.e. if $f, g \in \mathcal{R}([a,b])$ and $c \in \mathbb{R}$, then $f + cg \in \mathcal{R}([a,b])$.

Proof

Proof continued

Proposition (Rules for integration on $[a, b]$)

1. The constant function $f(x) = c$ is integrable and its integral is $c(b-a)$.
2. If f is Riemann integrable, then it is bounded.
3. If $f, g \in \mathcal{R}([a, b])$ and $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

4. If $f \in \mathcal{R}([a, b])$ and $g : [c, d] \to [a, b]$ is a continuously differentiable bijection with $g' > 0$, then

$$\int_a^b f(y)dy = \int_c^d f(g(x))g'(x)dx.$$

5. If $f, g : [a, b] \to \mathbb{R}$ are differentiable and $f', g' \in \mathcal{R}([a, b])$, then

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx.$$

Theorem (Fundamental Theorem of Calculus)

First part:

If $f : [a, b] \to \mathbb{R}$ is Riemann integrable then its indefinite integral

$$F(x) = \int_a^x f(t)dt$$

is a continuous function of x. In addition, the derivative of F exists and $F'(x) = f(x)$ at all $x \in [a, b]$ where f is continuous.

Second part:

Let $f : [a, b] \to \mathbb{R}$ and let F be a continuous function on $[a, b]$ with antiderivative f on (a, b), i.e. $F'(x) = f(x)$. Then if F is Riemann integrable on $[a, b]$,

$$\int_a^b f(x)dx = F(b) - F(a).$$

Drawbacks of the Riemann integral

- Riemann integration has many nice properies, but it also has some drawbacks
- To see this, we first introduce a nice alternative characterization of Riemann integrability
- Instead of looking at all the Riemann sums, we can restrict our attention to two special forms of the sum

Definition

Given a function $f\colon [a,b] \to \mathbb{R}$ and a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$, we define the *lower* and *upper sum* of f via

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i, \qquad U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i,$$

where $m_i = \inf\{f(t) \colon t \in [x_{i-1}, x_i]\}$ and $M_i = \sup\{f(t) \colon t \in [x_{i-1}, x_i]\}$. We define the *lower* and *upper integral* of f to be

$$\underline{I} = \sup_P L(f,P), \qquad \overline{I} = \inf_P U(f,P).$$

Since f is defined on a compact set, it will be bounded and hence the lower and upper integral exist and are finite.

One can think of the lower and upper integral as lower and upper bounds for the Riemann integral.

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a function. Then f is Riemann integrable if and only if $\underline{I} = \overline{I}$ and we have $\underline{I} = \overline{I} = I$.

A function that is not Riemann integrable

$$f : [0,1] \to \mathbb{R} : x \mapsto \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1 & \text{otherwise.} \end{cases}$$

Is this function Riemann integrable? Should it be integrable?

The End

References

Charles C. Pugh (2015). *Real Mathematical Analysis*. Undergraduate Texts in Mathematics. https://link-springer-com.myaccess.library.utoronto.ca/book/10.1007/978-3-319-17771-7