Exercises for Module 2: Set Theory

1. Is $\mathbb{R} \times \mathbb{R}$ with the ordering $(x_1, y_1) \preceq (x_2, y_2)$ if $x_1 \leq x_2$ a partially ordered set?

2. Let S be a non-empty set. A relation R on S is called an equivalence relation if it is

   (i) Reflexive: $(x, x) \in R$ for all $x \in S$

   (ii) Symmetric: if $(x, y) \in R$ then $(y, x) \in R$ for all $x, y \in S$

   (iii) Transitive: if $(x, y), (y, z) \in R$ then $(x, z) \in R$ for all $x, y, z \in S$

Given $x \in S$, the equivalence class of x (with respect to a given equivalence relation R) is defined to consist of those $y \in S$ for which $(x, y) \in R$. Show that two equivalence classes are either disjoint or identical.

3. Let $(X, \leq)$ be a partially ordered set and $S \subseteq X$ be bounded. Show that the infimum and supremum of S are unique (if they exist).

4. Let $S, T \subseteq \mathbb{R}$ and suppose both are bounded above. Define $S + T = \{s + t : \ s \in S, t \in T\}$. Show that $S + T$ is bounded above and $\sup(S + T) = \sup S + \sup T$.

5. Let $f : X \to Y$, $X, Y \subseteq \mathbb{R}$, be defined by the map $x \mapsto \sin(x)$. For what choices of X and Y is f injective, surjective, bijective, or neither?

6. Show that for sets $A, B \subseteq X$ and $f : X \to Y$, $f(A \cap B) \subseteq f(A) \cap f(B)$.

7. Let $f : X \to Y$ and $B \subseteq Y$. Prove that $f(f^{-1}(B)) \subseteq B$, with equality iff f is surjective.

8. Prove that $f(\cup_{i \in I} A_i) = \cup_{i \in I} f(A_i)$, where $f : X \to Y$, $A_i \subseteq X \, \forall i \in I$.