Exercises for Module 3: Set Theory II and Metric Spaces I

1. Show that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.

2. Show that $|(0,1)| = |(1,\infty)|$.

3. Show that the infinity norm $\|x\|_\infty$, $x \in \mathbb{R}^n$, is a norm.

4. Let (X, d) be any metric space, and define $\tilde{d} : X \times X \to \mathbb{R}$ by

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \quad x, y \in X.$$

Show that $\tilde{d}$ is a metric on X.

5. Let X be a set and define $d \colon X \times X \to \mathbb{R}$ by $d(x, x) = 0$ and $d(x, y) = 1$ for $x \neq y \in X$. Prove that d is a metric on X. What do open balls look like for different radii $r > 0$? What does an arbitrary open set look like?

6. Show that the infinite intersection of open sets may not be open and that the infinite union of closed sets may not be closed.