Module 3: Metric Spaces and Sequences I

Operational math bootcamp

Statistical Sciences
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Outline

- Finish cardinality section
- Metrics and norms
- Open and closed sets

Definition

Two sets A and B have same cardinality, $|A| = |B|$, if there exists bijection $f : A \to B$.

Example

Which is bigger, $\mathbb{N}$ or $\mathbb{N}_0$?

Cantor-Schröder-Bernstein

Definition

We say that the cardinality of a set A is less than the cardinality of a set B, denoted $|A| \leq |B|$ if there exists an injection $f : A \to B$.

Theorem (Cantor-Bernstein)

Let A, B, be sets. If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Example

$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$

Proof.

$\square$

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Definition

Let A be a set.

1. A is *finite* if there exists an $n \in \mathbb{N}$ and a bijection $f : \{1, \ldots, n\} \to A$
2. A is *countably infinite* if there exists a bijection $f : \mathbb{N} \to A$
3. A is *countable* if it is finite or countably infinite
4. A is *uncountable* otherwise

Example

The rational numbers are countable, and in fact $|\mathbb{Q}| = |\mathbb{N}|$.

Proof.

First we show $|\mathbb{N}| \leq |\mathbb{Q}^+|$. □

Proof.

Next, we show that $|\mathbb{Q}^+| \leq |\mathbb{N} \times \mathbb{N}|$.

Since we already proved $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$, this means $|\mathbb{N}| = |\mathbb{Q}^+|$. $\square$

Proof.

We can extend this to $\mathbb{Q}$ as follows:. $\square$

Theorem

The cardinality of $\mathbb{N}$ is smaller than that of $(0, 1)$.

Proof.

First, we show that there is an injective map from $\mathbb{N}$ to $(0, 1)$.

Next, we show that there is no surjective map from $\mathbb{N}$ to (0, 1). We use the fact that every number $r \in (0, 1)$ has a binary expansion of the form $r = 0.\sigma_1\sigma_2\sigma_3\ldots$ where $\sigma_i \in \{0, 1\}$, $i \in \mathbb{N}$. □

Proof.

Now we suppose in order to derive a contradiction that there does exist a surjective map f from $\mathbb{N}$ to (0, 1)., i.e. for $n \in \mathbb{N}$ we have $f(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\ldots$. This means we can list out the binary expansions, for example like

$$\begin{aligned} f(1) =& 0.00000000\ldots \\ f(2) =& 0.1111111111\ldots \\ f(3) =& 0.0101010101\ldots \\ f(4) =& 0.1010101010\ldots \end{aligned}$$

We will construct a number $\tilde{r} \in (0, 1)$ that is not in the image of f. □

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Proof.

Define $\tilde{r} = 0.\tilde{\sigma}_1\tilde{\sigma}_2\ldots$, where we define the nth entry of $\tilde{r}$ to be the the opposite of the nth entry of the nth item in our list:

$$\tilde{\sigma}_n = \begin{cases} 1 & \text{if } \sigma_n(n) = 0, \\ 0 & \text{if } \sigma_n(n) = 1. \end{cases}$$

Then $\tilde{r}$ differs from $f(n)$ at least in the nth digit of its binary expansion for all $n \in \mathbb{N}$. Hence, $\tilde{r} \notin f(\mathbb{N})$, which is a contradiction to f being surjective. This technique is often referred to as Cantor's diagonal argument. □

Proposition

(0,1) and $\mathbb{R}$ have the same cardinality.

Proof.

∎

We have shown that there are different sizes of infinity, as the cardinality of $\mathbb{N}$ is infinite but still smaller than that of $\mathbb{R}$ or $(0,1)$. In fact, we have

$$|\mathbb{N}| \quad |\mathbb{N}_0| \quad |\mathbb{Z}| \quad |\mathbb{Q}| \quad |\mathbb{R}|.$$

Because of this, there are special symbols for these two cardinalities: The cardinality of $\mathbb{N}$ is denoted $\aleph_0$, while the cardinality of $\mathbb{R}$ is denoted $\mathfrak{c}$.

Metric Spaces

Definition (Metric)

A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ that satisfies:

(a) Positive definiteness:

(b) Symmetry:

(c) Triangle inequality:

A set together with a metric is called a metric space.

Example ($\mathbb{R}^n$ with the Euclidean distance)

Definition (Norm)

A *norm* on an $\mathbb{F}$-vector space E is a function $\|\cdot\| : E \to \mathbb{R}$ that satisfies:

(a) Positive definiteness:

(b) Homogeneity:

(c) Triangle inequality:

A vector space with a norm is called a normed space. A normed space is a metric space using the metric $d(x, y) = \|x - y\|$.

Example (p-norm on $\mathbb{R}^n$)

The p-norm is defined for $p \geq 1$ for a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ as

The infinity norm is the limit of the p-norm as $p \to \infty$, defined as

Example (p-norm on $C([0,1];\mathbb{R})$)

If we look at the space of continuous functions $C([0,1];\mathbb{R})$, the p-norm is

and the $\infty-$norm (or sup norm) is

Definition

A subset A of a metric space (X, d) is *bounded* if there exists $M > 0$ such that $d(x, y) < M$ for all $x, y \in A$.

Definition

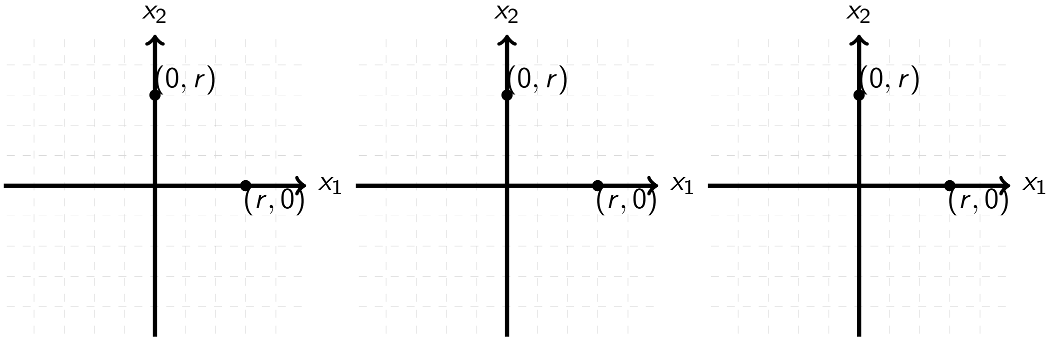
Let (X, d) be a metric space. We define the *open ball* centred at a point $x_0 \in X$ of radius $r > 0$ as

$$B_r(x_0) := \{x \in X : d(x, x_0) < r_0\}.$$

Example

In $\mathbb{R}$ with the usual norm (absolute value), open balls are symmetric open intervals, i.e.

Example: Open ball in $\mathbb{R}^2$ with different metrics



(a) 1-norm (taxicab metric) (b) 2-norm (Euclidean metric) (c) ∞-norm

Figure: $B_r(0)$ for different metrics

Definition (Open and closed sets)

Let (X, d) be a metric space.

- A set $U \subseteq X$ is *open* if for every $x \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$.
- A set $F \subseteq X$ is *closed* if $F^c := X \setminus F$ is open.

Proposition

Let (X, d) be a metric space.

1. *Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are open, then $A_1 \cap A_2$ is open.*
2. *If $A_i \subseteq X$, $i \in I$ are open, then $\cup_{i \in I} A_i$ is open.*

Proof.

(1) Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are open, then $A_1 \cap A_2$ is open.

(2) If $A_i \subseteq X$, $i \in I$ are open, then $\cup_{i \in I} A_i$ is open.

□

Using DeMorgan, we immediately have the following corollary:

Corollary

Let (X, d) be a metric space.

1. *Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are closed, then $A_1 \cup A_2$ is closed.*
2. *If $A_i \subseteq X$, $i \in I$ are closed, then $\cap_{i \in I} A_i$ is closed.*

Definition (Interior and closure)

Let $A \subseteq X$ where (X, d) is a metric space.

- The *closure* of A is $\overline{A} :=$
- The *interior* of A is $\mathring{A} :=$
- The *boundary* of A is $\partial A :=$

Example

Let $X = (a, b] \subseteq \mathbb{R}$ with the ordinary (Euclidean) metric. Then

Proposition

Let $A \subseteq X$ *where* (X, d) *is a metric space. Then* $\mathring{A} = A \setminus \partial A$.

Proof.

□

References

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