Module 3: Metric Spaces and Sequences I

Operational math bootcamp

Statistical Sciences
UNIVERSITY OF TORONTO

Emma Kroell

University of Toronto

July 14, 2022

Outline

- Finish cardinality section
- Metrics and norms
- Open and closed sets

Definition

Two sets A and B have same cardinality, $|A| = |B|$, if there exists bijection $f : A \to B$.

Example

Which is bigger, $\mathbb{N}$ or $\mathbb{N}_0$?

$$|\mathbb{N}| = |\mathbb{N}_0|$$

$$f : \mathbb{N}_0 \to \mathbb{N} \qquad n \mapsto n+1$$

is a bijection

Cantor-Schröder-Bernstein

Definition

We say that the cardinality of a set A is less than the cardinality of a set B, denoted $|A| \leq |B|$ if there exists an injection $f : A \to B$.

Theorem (Cantor-Bernstein)

Let A, B, be sets. If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Example

$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$

Proof.

First, we show $|\mathbb{N}| \le |\mathbb{N} \times \mathbb{N}|$.

$f: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ $\quad n \mapsto (n, 1)$ is an injection, therefore $|\mathbb{N}| \le |\mathbb{N} \times \mathbb{N}|$.

Next, we show $|\mathbb{N} \times \mathbb{N}| \le |\mathbb{N}|$.

$g: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ $\quad (n, m) \mapsto 2^n 3^m$

We claim g is an injection. Proof. Let $n_1, n_2, m_1, m_2 \in \mathbb{N}$ such that $2^{n_1} 3^{m_1} = 2^{n_2} 3^{m_2}$. By the Fund. Thm of Arithmetic, we must have $n_1 = n_2$ and $m_1 = m_2$. $\square$

$\therefore |\mathbb{N} \times \mathbb{N}| \le |\mathbb{N}|$.

Definition

Let A be a set.

1. A is *finite* if there exists an $n \in \mathbb{N}$ and a bijection $f : \{1, \ldots, n\} \to A$
2. A is *countably infinite* if there exists a bijection $f : \mathbb{N} \to A$
3. A is *countable* if it is finite or countably infinite
4. A is *uncountable* otherwise

Example

The rational numbers are countable, and in fact $|\mathbb{Q}| = |\mathbb{N}|$.

Proof.

First we show $|\mathbb{N}| \leq |\mathbb{Q}^+|$. $\mathbb{Q}^+ := \{ x \in \mathbb{Q} \mid x > 0 \}$

$$
\begin{array}{ccccc}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
2 & \cancel{\frac{2}{2}} & \frac{2}{3} & \cancel{\frac{2}{4}} & \cdots \\
3 & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}
$$

This is an injection from $\mathbb{N}$ to $\mathbb{Q}^+$.

$\square$

Proof.

Next, we show that $|\mathbb{Q}^+| \leq |\mathbb{N} \times \mathbb{N}|$.

$$f : \mathbb{Q}^+ \to \mathbb{N} \times \mathbb{N} \qquad \frac{p}{q} \to (p, q)$$

Since we already proved $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$, this means $|\mathbb{N}| = |\mathbb{Q}^+|$. $\square$

Proof.

We can extend this to $\mathbb{Q}$ as follows:.

Let $f : \mathbb{N} \to \mathbb{Q}^+$ be a bijection

Then define $g : \mathbb{N} \to \mathbb{Q}$ as follows

$$g(1) = 0$$

$$g(n) = \begin{cases} f(n) & n \text{ is even} \\ -f(n) & n \text{ is odd} \end{cases}$$

for $n > 1$.

Theorem

The cardinality of $\mathbb{N}$ is smaller than that of $(0, 1)$.

Proof.

First, we show that there is an injective map from $\mathbb{N}$ to $(0, 1)$.

$f : \mathbb{N} \to (0, 1) \qquad n \to \frac{1}{n}$

Next, we show that there is no surjective map from $\mathbb{N}$ to $(0,\ 1)$. We use the fact that every number $r \in (0, 1)$ has a binary expansion of the form $r = 0.\sigma_1\sigma_2\sigma_3\ldots$ where $\sigma_i \in \{0, 1\}$, $i \in \mathbb{N}$. $\square$

Proof.

Now we suppose in order to derive a contradiction that there does exist a surjective map f from $\mathbb{N}$ to $(0, 1)$., i.e. for $n \in \mathbb{N}$ we have $f(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\ldots$. This means we can list out the binary expansions, for example like

$$
\begin{aligned}
f(1) &= 0.00000000\ldots \\
f(2) &= 0.1111111111\ldots \\
f(3) &= 0.0101010101\ldots \\
f(4) &= 0.1010101010\ldots
\end{aligned}
$$

We will construct a number $\tilde{r} \in (0, 1)$ that is not in the image of f. □

Proof.

Define $\tilde{r} = 0.\tilde{\sigma}_1\tilde{\sigma}_2\ldots$, where we define the nth entry of $\tilde{r}$ to be the the opposite of the nth entry of the nth item in our list:

$$\tilde{\sigma}_n = \begin{cases} 1 & \text{if } \sigma_n(n) = 0, \\ 0 & \text{if } \sigma_n(n) = 1. \end{cases}$$

Then $\tilde{r}$ differs from $f(n)$ at least in the nth digit of its binary expansion for all $n \in \mathbb{N}$. Hence, $\tilde{r} \notin f(\mathbb{N})$, which is a contradiction to f being surjective. This technique is often referred to as Cantor's diagonal argument. $\square$

Proposition

(0,1) and $\mathbb{R}$ have the same cardinality.

Proof.

$$f : \mathbb{R} \to (0,1) \quad x \mapsto \frac{1}{\pi}\left(\arctan(x) + \frac{\pi}{2}\right)$$

is a bijection

$\square$

We have shown that there are different sizes of infinity, as the cardinality of $\mathbb{N}$ is infinite but still smaller than that of $\mathbb{R}$ or $(0,1)$. In fact, we have

$$|\mathbb{N}| = |\mathbb{N}_0| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|.$$

Because of this, there are special symbols for these two cardinalities: The cardinality of $\mathbb{N}$ is denoted $\aleph_0$, while the cardinality of $\mathbb{R}$ is denoted $\mathfrak{c}$.

Metric Spaces

Definition (Metric)

A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ that satisfies:

(a) Positive definiteness: $d(x,y) \geq 0 \quad \forall x,y \in X$ & $d(x,y) = 0 \iff x = y$

(b) Symmetry: $x,y \in X$, $d(x,y) = d(y,x)$

(c) Triangle inequality: $x,y,z \in X \quad d(x,z) \leq d(x,y) + d(y,z)$

A set together with a metric is called a metric space.

Example ($\mathbb{R}^n$ with the Euclidean distance)

$$d(x,y) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2} \quad \text{for } x, y \in \mathbb{R}^n$$

$\mathbb{R}^n$ with Euclidean distance is a metric space

$\mathbb{F}$: field

$\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$

Definition (Norm)

A *norm* on an $\mathbb{F}$-vector space E is a function $\|\cdot\| : E \to \mathbb{R}$ that satisfies:

(a) Positive definiteness: $\|x\| \geq 0 \ \forall x \in E$ & $\|x\| = 0 \Leftrightarrow x = 0$

(b) Homogeneity: $x \in E$ $\alpha \in \mathbb{F}$, $\|\alpha x\| = |\alpha| \|x\|$

(c) Triangle inequality: $x, y \in E$ $\|x+y\| \leq \|x\| + \|y\|$

A vector space with a norm is called a normed space. A normed space is a metric space using the metric $d(x, y) = \|x - y\|$.

Example (p-norm on $\mathbb{R}^n$)

The p-norm is defined for $p \geq 1$ for a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ as

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

The infinity norm is the limit of the p-norm as $p \to \infty$, defined as

$$\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$$

Example (p-norm on $C([0,1];\mathbb{R})$)

If we look at the space of continuous functions $C([0,1];\mathbb{R})$, the p-norm is

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p}$$

and the ∞−norm (or sup norm) is

$$\|f\|_\infty = \max_{x \in [0,1]} |f(x)|$$

Statistical Sciences
UNIVERSITY OF TORONTO

Definition

A subset A of a metric space (X, d) is *bounded* if there exists $M > 0$ such that $d(x, y) < M$ for all $x, y \in A$.

Definition

Let (X, d) be a metric space. We define the *open ball* centred at a point $x_0 \in X$ of radius $r > 0$ as

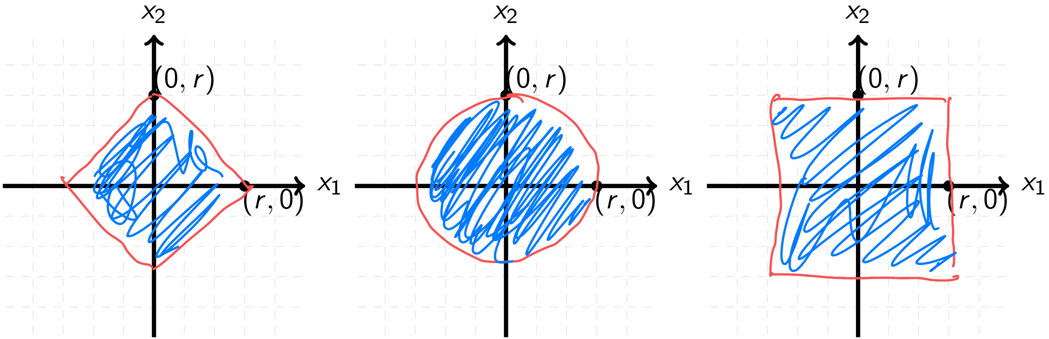
$$B_r(x_0) := \{x \in X : d(x, x_0) < r\}.$$

Example

In $\mathbb{R}$ with the usual norm (absolute value), open balls are symmetric open intervals, i.e.

$$B_r(x_0) = (x_0 - r, x_0 + r)$$

Example: Open ball in $\mathbb{R}^2$ with different metrics



(a) 1-norm (taxicab metric) (b) 2-norm (Euclidean metric) (c) ∞-norm

$d(x,y) = \sum_{i=1}^{2} |x_i - y_i|$

$\max_{j=1,2} |x_j - y_j|$

Figure: $B_r(0)$ for different metrics

Definition (Open and closed sets)

Let (X, d) be a metric space.

- A set $U \subseteq X$ is *open* if for every $x \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$.
- A set $F \subseteq X$ is *closed* if $F^c := X \setminus F$ is open.

$\emptyset$, X are both open & closed

Proposition

Let (X, d) be a metric space.

1. *Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are open, then $A_1 \cap A_2$ is open.*
2. *If $A_i \subseteq X$, $i \in I$ are open, then $\cup_{i \in I} A_i$ is open.*

Proof.

(1) Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are open, then $A_1 \cap A_2$ is open.

Since A_1 is open, for each $x \in A_1$, $\exists \varepsilon_1 > 0$ s.t. $B_{\varepsilon_1}(x) \subseteq A_1$. Since A_2 is open, $\exists \varepsilon_2 > 0$ s.t. $B_{\varepsilon_2}(x) \subseteq A_2$.

Let $x \in A_1 \cap A_2$. Choose $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then $B_\varepsilon(x) \subseteq A_1 \cap A_2$.

(2) If $A_i \subseteq X$, $i \in I$ are open, then $\cup_{i \in I} A_i$ is open.

Let $x \in \bigcup_{i \in I} A_i$. $\exists i \in I$ s.t. $x \in A_i$.

Since A_i is open, $\exists \varepsilon > 0$ s.t $B_\varepsilon(x) \subseteq A_i$.

Since $A_i \subseteq \bigcup_{i \in I} A_i$, we are done.

$\square$

Using DeMorgan, we immediately have the following corollary:

Corollary

Let (X, d) be a metric space.

1. *Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are closed, then $A_1 \cup A_2$ is closed.*
2. *If $A_i \subseteq X$, $i \in I$ are closed, then $\cap_{i \in I} A_i$ is closed.*

Definition (Interior and closure)

Let $A \subseteq X$ where (X, d) is a metric space.

- The *closure* of A is $\overline{A} := \{x \in X : \forall \varepsilon > 0 \quad B_\varepsilon(x) \cap A \neq \emptyset\}$
- The *interior* of A is $\mathring{A} := \{x \in A : \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq A\}$
- The *boundary* of A is $\partial A := \{x \in X : \forall \varepsilon > 0,\ B_\varepsilon(x) \cap A \neq \emptyset \text{ and } B_\varepsilon(x) \cap A^c \neq \emptyset\}$

Example

Let $X = (a, b] \subseteq \mathbb{R}$ with the ordinary (Euclidean) metric. Then

$\overline{X} = [a,b]$, $\mathring{X} = (a,b)$, $\partial X = \{a,b\}$

Proposition

Let $A \subseteq X$ where (X, d) is a metric space. Then $\mathring{A} = A \setminus \partial A$.

Proof.

First, show $\mathring{A} \subseteq A \setminus \partial A$.

Let $x \in \mathring{A}$. $\exists \varepsilon > 0$ s.t $B_\varepsilon(x) \subseteq A$. Clearly $x \in A$.

$\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \cap A^c = \emptyset$. Therefore $x \notin \partial A$.

$\therefore x \in A \setminus \partial A$

Next, we show, $A \setminus \partial A \subseteq \mathring{A}$.

Let $x \in A \setminus \partial A$. Then $x \in A$ and $x \notin \partial A$.

$x \notin \partial A$ means $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \cap A = \emptyset$ or

$B_\varepsilon(x) \cap A^c = \emptyset$. Since $x \in A$, the first is false, $\square$

so $\exists \varepsilon > 0$ s.t $B_\varepsilon(x) \cap A^c = \emptyset$. $\Rightarrow B_\varepsilon(x) \subseteq A$.

References

Thus $x \in \mathring{A}$.

Runde ,Volker (2005). *A Taste of Topology*. Universitext. url: https://link.springer.com/book/10.1007/0-387-28387-0

Zwiernik, Piotr (2022). *Lecture notes in Mathematics for Economics and Statistics*. url: http://84.89.132.1/ piotr/docs/RealAnalysisNotes.pdf