# Module 5: Metric spaces & Topology

## Operational math bootcamp

Statistical Sciences
UNIVERSITY OF TORONTO

Emma Kroell

University of Toronto

July 19, 2022

# Last time

Finished our discussion of open and closed sets:

- Introduced a cluster points of a set:
  $x \in X$ is a *cluster point* of $A$ if for every $\epsilon > 0$, $B_\epsilon(x)$ contains **infinitely** many points in $A$.
- Sequence characterization of a closed set:
  A set $F \subseteq X$, where $(X, d)$ is a metric space, is closed if and only if every sequence in $F$ which converges in $X$ converges to a point in $F$.

# Last time

Discussed sequences, which includes Cauchy sequences and subsequences: in $\mathbb{R}$: $|x_n - x| < \varepsilon$

- Convergent sequence: $x_n \to x \Leftrightarrow \forall \epsilon > 0 \; \exists \, n_\epsilon \in \mathbb{N}$ s.t. $d(x_n, x) < \epsilon$ for all $n \geq n_\epsilon$
- Cauchy sequence: $\forall \epsilon > 0 \; \exists \, n_\epsilon \in \mathbb{N}$ s.t. $d(x_n, x_m) < \epsilon$ for all $n, m \geq n_\epsilon$
- Proved that a convergent sequence is Cauchy
- Discussed complete metric spaces, where all Cauchy sequences converge (like $\mathbb{R}$ with the usual metric, absolute value)
- Proved that a sequence converges to $x$ if and only if all subsequences converge to $x$

# Last time

Discussed continuous functions:

- Showed that these three definitions of continuous are equivalent in metric spaces:

  $f : X \to Y$ is *continuous* where $(X, d_X)$ and $(Y, d_Y)$ are metric spaces $\Leftrightarrow$
  - if for every $x_0 \in X$, for every sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ that converges to $x_0$, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.
  - if for every $x_0 \in X$, for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0))) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$

    1 1 on $\mathbb{R}$
  - if $U \subseteq Y$ is open, then $f^{-1}(U)$ is open

- Briefly discussed other types of continuity (uniform, Lipschitz) and the Contraction Mapping Theorem

Statistical Sciences
UNIVERSITY OF TORONTO

# Outline for today

- Finish metric spaces
  - Equivalent metrics
  - A few extra topics on $\mathbb{R}$, including lim sup and lim inf
- Start topology
  - Basic definitions

# Equivalent metrics

**Definition (Equivalent metrics)**

Two metrics $d_1$ and $d_2$ on a set $X$ are *equivalent* if the identity maps from $(X, d_1)$ to $(X, d_2)$ and from $(X, d_2)$ to $(X, d_1)$ are continuous.

$1_X : x \mapsto x$

**Proposition**

Two metrics $d_1$, $d_2$ on a set $X$ are equivalent if and only if they have the same open sets or the same closed sets.

## Definition

Two metrics $d_1$ and $d_2$ on a set $X$ are *strongly equivalent* if for every $x, y \in X$, there exists constants $\alpha > 0$ and $\beta > 0$ such

$$\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y).$$

If two metrics are strongly equivalent then they are equivalent. The proof of this is one of the exercises.

## Example

We show that the Euclidean distance (induced by 2-norm) and the metric induced by the $\infty$-norm are equivalent on $\mathbb{R}^n$.

$$\|x-y\|_2 = \sqrt{\sum_{j=1}^{n}(x_j-y_j)^2}, \quad \|x-y\|_\infty = \max_{j=1,\dots,n}|x_j-y_j|$$

$$\|x-y\|_2 = \sqrt{\sum_{j=1}^{n}(x_j-y_j)^2} \le \sqrt{n \max_{j=1,\dots,n}(x_j-y_j)^2} = \sqrt{n}\max_{j=1,\dots,n}|x_j-y_j|$$

$$= \sqrt{n}\,\|x-y\|_\infty$$

$$\|x-y\|_\infty = \max_{j=1,\dots,n}|x_j-y_j| = \sqrt{\max_{j=1,\dots,n}(x_j-y_j)^2} \le \sqrt{\sum_{j=1}^{n}(x_j-y_j)^2}$$

$$= \|x-y\|_2$$

Can you think of an example that we've seen of a metric that isn't equivalent to the Euclidean metric?

# Right and left continuous

Recall: $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|x_0 - y| < \delta$ implies $|f(x_0) - f(y)| < \epsilon$.

$\iff \delta < x_0 - y < \delta \implies x_0 - \delta < y < x_0 + \delta$

## Definition

Let $f : \mathbb{R} \to \mathbb{R}$.

- $f$ is *left continuous* at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$, such $|f(x_0) - f(x)| < \epsilon$ whenever $x_0 - \delta < x < x_0$.
- $f$ is *right continuous* at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$, such $|f(x_0) - f(x)| < \epsilon$ whenever $x_0 < x < x_0 + \delta$.

We say that $f$ is left continuous if it is left continuous at all points in the domain, and similar for right continuous.

## Proposition

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if it is left and right continuous.

## Proof.

($\Leftarrow$) Suppose $f$ is left & right continuous. Let $\varepsilon > 0$ arbitrary.
By the def of left-cont., $\exists\, \delta_1 > 0$ s.t. $|f(x_0) - f(x)| < \varepsilon$ whenever $x_0 - \delta_1 < x < x_0$. By def of right-cont, $\exists\, \delta_2 > 0$ s.t. $|f(x_0) - f(x)| < \varepsilon$ whenever $x_0 < x < x_0 + \delta_2$.
Choose $\delta = \min\{\delta_1, \delta_2\}$. Let $x \in \mathbb{R}$ s.t. $|x - x_0| < \delta$.
$\Rightarrow x_0 - \delta < x < x_0 + \delta$.

<u>Case 1:</u> suppose $x < x_0$. Then $x_0 - \delta_1 < x < x_0$
$\Rightarrow |f(x) - f(x_0)| < \varepsilon$ by left-cont

□

## Proof continued

Case 2: suppose $x > x_0$. Then $x_0 < x < x_0 + \delta_2$.

$\Rightarrow |f(x) - f(x_0)| < \varepsilon$ by right-cont

(If $x = x_0$, this is trivial).

$\Rightarrow$ Let $f: \mathbb{R} \to \mathbb{R}$ be continuous. This means $\forall \varepsilon > 0$ $\exists \delta_0 > 0$ s.t. $|x - x_0| < \delta_0 \Rightarrow |f(x) - f(x_0)| < \varepsilon$

Let $\varepsilon > 0$ be arbitrary. Choose $\delta = \delta_0$.

Let $x \in \mathbb{R}$ s.t. $x_0 \leq x < x_0 + \delta_0$. $\therefore f$ is right-cont

$\Rightarrow x_0 - \delta_0 < x < x_0 + \delta_0$

$\Rightarrow |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

# Bounded sequences and monotone convergence

## Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. We call $(x_n)_{n\in\mathbb{N}}$ *bounded* if there exists an $M > 0$ such that $|x_n| < M$ for all $n \in \mathbb{N}$.

## Theorem (Monotone convergence theorem)

non-decreasing

(i) *Suppose* $(x_n)_{n\in\mathbb{N}}$ *is an increasing sequence, i.e.* $x_n \leq x_{n+1}$ *for all* $n \in \mathbb{N}$*, and that it is bounded (above). Then the sequence converges. Furthermore,* $\lim_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} x_n$*, where* $\sup_{n\in\mathbb{N}} x_n := \sup\{x_n : n \in \mathbb{N}\}$.

(ii) *Suppose* $(x_n)_{n\in\mathbb{N}}$ *is a decreasing sequence, i.e.* $x_n \geq x_{n+1}$ *for all* $n \in \mathbb{N}$*, which is bounded (below). Then the sequence converges and* $\lim_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} x_n := \inf\{x_n : n \in \mathbb{N}\}$.

Convention: $\sup A = \infty$ if $A \subseteq \mathbb{R}$ is not bounded above and $\inf A = -\infty$ if $A$ is not bounded below.

**Lemma**

*If $A \subseteq B \subseteq \mathbb{R}$ is non-empty, then* $\inf A \leq \sup A$, $\sup A \leq \sup B$, *and* $\inf A \geq \inf B$.

The proof of this follows from the definition of greatest lower and least upper bound.

## Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. We define the *limit superior* of $(x_n)_{n\in\mathbb{N}}$ as

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \sup_{k\geq n} x_k.$$

Similarly we define the *limit inferior* of $(x_n)_{n\in\mathbb{N}}$ as

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} \inf_{k\geq n} x_k.$$

If the sequence $(x_n)_{n\in\mathbb{N}}$ is not bounded above, then $\limsup_{n\to\infty} x_n = \infty$. Similarly, if the sequence $(x_n)_{n\in\mathbb{N}}$ is not bounded below, then $\liminf_{n\to\infty} x_n = -\infty$.

## Proposition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$.

- The sequence of suprema, $s_n = \sup_{k\geq n} x_k$, is decreasing and the sequence of infima, $i_n = \inf_{k\geq n} x_k$, is increasing.
- The limit superior and the limit inferior of a bounded sequence always exist and are finite.

## Proof.

The first part is true by the previous Lemma. The second follows by the Monotone Convergence Theorem. $\square$

## Theorem

*Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then the sequence converges to $x \in \mathbb{R}$ if and only if* $\limsup_{n\to\infty} x_n = x = \liminf_{n\to\infty} x_n$.

## Proof.

Notation: $i_n := \inf_{k\geq n} x_k$ & $s_n := \sup_{k\geq n} x_k$, $n\in\mathbb{N}$

$(\Rightarrow)$ Suppose $x_n \to x$, $x\in\mathbb{R}$. Let $\varepsilon > 0$. By definition of $x_n \to x$, $\exists N \in \mathbb{N}$ s.t for all $n \geq N$, $|x - x_n| < \varepsilon$, i.e. $x-\varepsilon < x_n < x+\varepsilon$

$x-\varepsilon < x_n$ $\forall n \geq N$, $x-\varepsilon$ is a lower bound for the set $\{x_n : n \geq N\}$ $\Rightarrow$ $x-\varepsilon \leq i_N$

Similarly, $x_n < x+\varepsilon$ $\forall n \geq N$, so $s_N \leq x+\varepsilon$.

## Proof continued

monotone converge thm

$$x - \varepsilon \le i_N \le \lim_{n\to\infty} i_n = \liminf_{n\to\infty} x_n$$

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} s_n \le s_N < x + \varepsilon$$

$i_n \le s_n \; \forall n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} i_n \le \lim_{n\to\infty} s_n$

$$\underset{\substack{\shortparallel \\ \liminf_{n\to\infty} x_n}}{\lim_{n\to\infty} i_n} \le \underset{\substack{\shortparallel \\ \limsup_{n\to\infty} x_n}}{\lim_{n\to\infty} s_n}$$

$$\therefore\; x - \varepsilon \le \liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n \le x + \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$.

# Proof continued

($\Leftarrow$) Suppose $\limsup_{n\to\infty} x_n = x = \liminf_{n\to\infty} x_n$.

Let $\varepsilon > 0$.

Since $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} s_n = x$, $\exists N_1 \in \mathbb{N}$ s.t

$|s_n - x| < \varepsilon \quad \forall n \geq N_1$.

$\Rightarrow x_k \leq s_{N_1} < x + \varepsilon \quad \forall k \geq N_1$

Similarly, $\exists N_2 \in \mathbb{N}$ s.t. $|i_n - x| < \varepsilon \quad \forall n \geq N_2$,

so $x - \varepsilon < i_{N_2} < x_k \quad \forall k \geq N_2$

Take $N = \max\{N_1, N_2\}$. Then $x - \varepsilon < x_k < x + \varepsilon$

$\forall k \geq N$, so $x_n \to x$.

We can extend this easily to a sequence of functions $f_n \colon X \to \mathbb{R}$ as follows:

Define $f = \limsup_{n\to\infty} f_n$ to be the function defined pointwise by $f(x) = \limsup_{n\to\infty}(f_n(x))$ and similar for the limit inferior.

There also exists a set theoretic version in terms of unions and intersections which you will encounter in probability.

# Topology

- Let $X$ be a set. If $X$ is not a metric space, can we still have open and closed sets?
- One can think of a topology on $X$ as a specification of what subsets of $X$ are open

## Definition

Let $\mathcal{T} \subseteq \mathcal{P}(X)$. We call $\mathcal{T}$ a *topology* on $X$ if the following holds:

(i) $\emptyset, X \in \mathcal{T}$

(ii) Let $A$ be an arbitrary index set. If $U_\alpha \in \mathcal{T}$ for $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$ ($\mathcal{T}$ is closed under taking arbitrary unions)

(iii) Let $n \in \mathbb{N}$. If $U_1, \ldots, U_n \in \mathcal{T}$, then $\bigcap_{i=1}^n U_i \in \mathcal{T}$ ($\mathcal{T}$ is closed under taking finite intersections)

If $U \in \mathcal{T}$, we call $U$ *open*. We call $U \subseteq X$ *closed*, if $U^c \in \mathcal{T}$. We call $(X, \mathcal{T})$ a *topological space*.

### Example

For a set $X$, the following $\mathcal{T} \subseteq \mathcal{P}(X)$ are examples of topologies on $X$.

- Trivial topology: $\mathcal{T} = \{\emptyset, X\}$,
- Discrete topology: $\mathcal{T} = \mathcal{P}(X)$,
- Let $X$ be an infinite set. Then, $\mathcal{T} = \{U \subseteq X : U^c \text{ is finite}\} \cup \emptyset$ defines a topology on $X$.
- Topology induced by a metric: i.e. if $d$ is a metric on $X$ we can define

$$\mathcal{T}_d = \{U \subseteq X \mid \forall x \in U\, \exists \epsilon > 0 \text{ such that } B_\epsilon(x) \subseteq U\}.$$

The discrete topology is also induced by a metric, can you guess which one?

Given a topological space $(X, \mathcal{T})$ and a subset $Y \subseteq X$, we can restrict the topology on $X$ to $Y$ which leads to the next definition.

### Definition (Relative topology)

Given a topological space $(X, \mathcal{T})$ and an arbitrary non-empty subset $Y \subseteq X$, we define the relative topology on $Y$ as follows

$$\mathcal{T}|_Y = \{U \cap Y : U \in \mathcal{T}\}.$$

## Definition

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$ be any subset.

- The *interior* of $A$ is $\mathring{A} := \{a \in A \colon \exists U \in \mathcal{T} \text{ s.t. } U \subseteq A \text{ and } a \in U\}$.
- The *closure* of $A$ is $\overline{A} := \{x \in X : \forall U \in \mathcal{T} \text{ with } x \in U, U \cap A \neq \emptyset\}$.
- The *boundary* of $A$ is
  $\partial A := \{x \in X : \forall U \in \mathcal{T} \text{ with } x \in U,\ U \cap A \neq \emptyset \text{ and } U \cap A^c \neq \emptyset\}$.

Statistical Sciences
UNIVERSITY OF TORONTO

- The *interior* of $A$ is $\mathring{A} := \{a \in A\colon \exists U \in \mathcal{T} \text{ s.t. } U \subseteq A \text{ and } a \in U\}$.
- The *closure* of $A$ is $\overline{A} := \{x \in X\colon \forall U \in \mathcal{T} \text{ with } x \in U, U \cap A \neq \emptyset\}$.
- The *boundary* of $A$ is $\partial A := \{x \in X\colon \forall U \in \mathcal{T} \text{ with } x \in U,\ U \cap A \neq \emptyset \text{ and } U \cap A^c \neq \emptyset\}$.

## Example

Let $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then

- $\mathring{\{a\}} = \{a\}$
- $\mathring{\{c\}} = \emptyset$
- $\overline{\{a\}} = \{a, c\}$
- $\overline{\{c\}} = \{c\}$

$a \in \{a\}, \{a, b\}, X$

$b \in \{b\}, \{a, b\}, X$

$c \in X$

## Proposition

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then,

$$\overline{A} = \underbrace{\bigcap\{F : F \text{ is closed and } A \subseteq F\}}_{:= A'}.$$

## Proof.

$\overline{A} \subseteq A'$: We show the contrapositive, i.e. $(A')^c \subseteq (\overline{A})^c$.
Let $x \in (A')^c$. Then $x \notin A'$. $(A')^c$ is open, and in fact it is an open set that contains $x$.
Since $A \subseteq A'$, $A \cap (A')^c = \emptyset$. $\therefore x \notin \overline{A}$ i.e. $x \in \overline{A}^c$

$\square$

## Proof continued

$A' \subseteq \bar{A}$. We prove the contrapositive.

$(\bar{A})^c \subseteq (A')^c$. Let $x \in (\bar{A})^c$. Then $x \notin \bar{A}$.

Then $\exists\, U$ open with $x \in U$ and $U \cap A = \emptyset$.

$\Rightarrow A \subseteq U^c$. We know that $U^c$, by def,

$A' \subseteq U^c$. Since $x \notin U^c$, $x \notin A'$.

$\therefore A' \subseteq \bar{A}$

Similarly, one can show $\mathring{A} = \bigcup\{U : U \text{ is open and } U \subseteq A\}$. Hence, we see that the interior of $A$ is the largest open set contained in $A$ and the closure is the smallest closed set that contains $A$.

# Next time

- Finish topology
  - Dense subsets
  - Compactness
  - Continuity
- Start linear algebra
  - Vector spaces and subspaces

# References

Jiří Lebl (2022). *Basic Analysis I*. Vol. 1. Introduction to Real Analysis. https://www.jirka.org/ra/realanal.pdf

Runde, Volker (2005). *A Taste of Topology*. Universitext. url: https://link.springer.com/book/10.1007/0-387-28387-0