# Module 6: End of Topology, Start of Linear Algebra

## Operational math bootcamp

Statistical Sciences
UNIVERSITY OF TORONTO

Emma Kroell

University of Toronto

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# Outline

- Finish topology
  - Dense subsets
  - Compactness
  - Continuity
- Start linear algebra
  - Vector spaces and subspaces

# Last time

## Definition (Topology)

Let $\mathcal{T} \subseteq \mathcal{P}(X)$. We call $\mathcal{T}$ a *topology* on $X$ if the following holds:

(i) $\emptyset, X \in \mathcal{T}$

(ii) Let $A$ be an arbitrary index set. If $U_\alpha \in \mathcal{T}$ for $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$ ($\mathcal{T}$ is closed under taking arbitrary unions)

(iii) Let $n \in \mathbb{N}$. If $U_1, \dots, U_n \in \mathcal{T}$, then $\bigcap_{i=1}^n U_i \in \mathcal{T}$ ($\mathcal{T}$ is closed under taking finite intersections)

If $U \in \mathcal{T}$, we call $U$ *open*. We call $U \subseteq X$ *closed*, if $U^c \in \mathcal{T}$. We call $(X, \mathcal{T})$ a *topological space*.

## Definition

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$ be any subset.

- The *interior* of $A$ is $\mathring{A} := \{a \in A \colon \exists U \in \mathcal{T} \text{ s.t. } U \subseteq A \text{ and } a \in U\}$.
- The *closure* of $A$ is $\overline{A} := \{x \in X \colon \forall U \in \mathcal{T} \text{ with } x \in U, U \cap A \neq \emptyset\}$.
- The *boundary* of $A$ is $\partial A := \{x \in X \colon \forall U \in \mathcal{T} \text{ with } x \in U,\ U \cap A \neq \emptyset \text{ and } U \cap A^c \neq \emptyset\}$.

# Density

**Definition**

Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is called *dense* if $\overline{A} = X$.

Using the definition of closure, we see that $A \subseteq X$ is dense if and only if for all non-empty $U \in \mathcal{T}$, $U \cap A \neq \emptyset$.

**Example**

- The rationals $\mathbb{Q}$ are dense in the reals $\mathbb{R}$.
- The only dense subset in $(X, \mathcal{P}(X))$ is $X$ itself.
- Any non-empty subset is dense in $(X, \{\emptyset, X\})$.

# Separability

**Definition**

A topological space $(X, \mathcal{T})$ is *separable* if it contains a countable dense subset.

**Example**

# Hausdorff space

**Definition**

A topological space $(X, \mathcal{T})$ is called *Hausdorff* if for all $x \neq y \in X$ there exist open sets $U_x, U_y$ with $x \in U_x$ and $y \in U_y$ such that $U_x \cap U_y = \emptyset$.

So in a Hausdorff space, we can separate any two elements using open sets.

## Example

Let $(X, d)$ be a metric space. Then $(X, \mathcal{T}_d)$ is Hausdorff, where $\mathcal{T}_d$ is the topology induced by the metric $d$.

## Example

Let $X$ be an infinite set and $\mathcal{T} = \{U \subseteq X \colon U^c \text{ is finite}\} \cup \emptyset$. Then $(X, \mathcal{T})$ is not Hausdorff.

# Compactness

**Definition**

Let $(X, \mathcal{T})$ be a topological space and $K \subseteq X$.

A collection $\{U_i\}_{i \in I}$ of open sets is called *open cover* of $K$ if $K \subseteq \cup_{i \in I} U_i$.

The set $K$ is called *compact* if for all open covers $\{U_i\}_{i \in I}$ there exists a finite subcover, meaning there exists an $n \in \mathbb{N}$ and $\{U_1, \ldots, U_n\} \subseteq \{U_i\}_{i \in I}$ such that $K \subseteq \cup_{i=1}^{n} U_i$.

## Example

Let $S \subseteq X$ where $(X, \mathcal{T})$ is a topological space. If $S$ is finite, then it is compact.

## Example

$(0, 1)$ is not compact.

### Proposition

Let $(X, \mathcal{T})$ be a topological space and take a non-empty subset $K \subseteq X$. The following holds:

1. If $X$ is compact and $K$ is closed, then $K$ is compact (i.e. closed subsets of compact sets are compact).
2. If $(X, \mathcal{T})$ is Hausdorff, then $K$ being compact implies that $K$ is closed.

## Proof.

(1) If $X$ is compact and $K \subseteq X$ is closed, then $K$ is compact

$\square$

## Proof.

(2) If $(X, \mathcal{T})$ is Hausdorff, then $K \subseteq X$ compact $\Leftrightarrow$ $K$ is closed.

$\square$

## Proof continued

# Compactness on $\mathbb{R}^n$

**Theorem (Heine-Borel Theorem)**

*Let $K \subseteq \mathbb{R}^n$. Then $K$ is compact with respect to the topology induced by the Euclidean distance if and only if it is closed and bounded.*

Just as we had a sequential characterization of the closure of a set in metric spaces, we similarly have a sequential characterization of compactness.

**Theorem**

*Let $(X, d)$ be a metric space. Then $K \subset X$ is compact with respect to the metric induced by $d$ if and only if every sequence in $K$ admits a subsequence converging to some point in $K$.*

A corollary of this statement together with Heine-Borel is the Bolzano-Weierstrass theorem.

**Corollary (Bolzano-Weierstrass)**

*Any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

# Continuity on a topological space

**Definition**

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f\colon X \to Y$ is called *continuous* if for all $U \in \mathcal{T}_Y$, $f^{-1}(U) \in \mathcal{T}_X$, i.e. the pre-image of open sets is open.

We can also specify continuity at a point $x_0 \in X$.

**Definition**

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f\colon X \to Y$ is called *continuous at $x_0 \in X$* if for all $U \in \mathcal{T}_Y$ with $f(x_0) \in U$, $f^{-1}(U) \in \mathcal{T}_X$, i.e. the preimage of open sets containing $f(x_0)$ is open (and contains $x_0$).

## Proposition

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Suppose $K \subset X$ is compact and let $f : K \to Y$ be continuous. Then $f(K)$ is compact.

Recall from the set theory section:
If $f : X \to Y$:

1. $A \subseteq B \subseteq Y \Rightarrow f^{-1}(A) \subseteq f^{-1}(B)$ and $A \subseteq B \subseteq X \Rightarrow f(A) \subseteq f(B)$
2. $f^{-1}(\cup_{i \in I} A_i) = \cup_{i \in I} f^{-1}(A_i)$, where $A_i \subseteq Y \; \forall i \in I$
3. $f(\cup_{i \in I} A_i) = \cup_{i \in I} f(A_i)$, where $A_i \subseteq X \; \forall i \in I$
4. $A \subseteq X \Rightarrow A \subseteq f^{-1}(f(A))$
5. $B \subseteq Y \Rightarrow f(f^{-1}(B)) \subseteq B$

## Proof.

## Proof continued

# Linear Algebra

## Definition

We call $V$ a **vector space** if the following hold:

(A) *Commutativity in addition:* $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$

(B) *Associativity in addition:* $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$

(C) *Existence of a neutral element, addition:* There exists a vector $\mathbf{0}$ such that for any $\mathbf{v} \in V$, $\mathbf{0} + \mathbf{v} = \mathbf{v}$

(D) *Additive inverse:* For every $\mathbf{v} \in V$, there exists another vector, which we denote $-\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

(E) *Existence of a neutral element, multiplication:* For any $\mathbf{v} \in V$, $1 \times \mathbf{v} = \mathbf{v}$

(F) *Associativity in multiplication:* Let $\alpha, \beta \in \mathbb{F}$. For any $\mathbf{v} \in V$, $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$

(G) Let $\alpha \in \mathbb{F}, \mathbf{u}, \mathbf{v} \in V$. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.

(H) Let $\alpha, \beta \in \mathbb{F}, \mathbf{v} \in V$. $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$.

Elements of the vector space are called vectors.
Most often we will assume $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.

### Example

The following are vector spaces:

- $\mathbb{R}^n$
- $\mathbb{C}^n$
- $C(\mathbb{R}; \mathbb{R})$, continuous functions from $\mathbb{R}$ to $\mathbb{R}$
- $M_{n \times m}$, matrices of size $n \times m$
- $\mathbb{P}_n$ (polynomials of degree $n$, $p(x) = a_0 + a_1 x + \ldots + a_n x^n$).

## Lemma

For every $\mathbf{v} \in V$, $0\mathbf{v} = \mathbf{0}$.

## Proof.

$\square$

**Lemma**

For every $\mathbf{v} \in V$, we have $-\mathbf{v} = (-1) \times \mathbf{v}$.

**Proof.**

$\square$

## Definition

A subset $U$ of $V$ is called a **subspace** of of $V$ if $U$ is also a vector space (using the same addition and scalar multiplication as on $V$).

## Proposition

A subset $U$ of $V$ is a subspace of $V$ if and only if $U$ satisfies the following three conditions:

1. $\mathbf{0} \in U$
2. Closed under addition: $\mathbf{u}, \mathbf{v} \in U$ implies $\mathbf{u} + \mathbf{v} \in U$
3. Closed under scalar multiplication: $\alpha \in \mathbb{F}$ and $\mathbf{u} \in U$ implies $\alpha \mathbf{u} \in U$

### Proof.

$(\Rightarrow)$

$(\Leftarrow)$

$\square$

# References


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