Module 6: End of Topology, Start of Linear Algebra

Operational math bootcamp

Statistical Sciences
UNIVERSITY OF TORONTO

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Outline

- Finish topology
  - Dense subsets
  - Compactness
  - Continuity
- Start linear algebra
  - Vector spaces and subspaces

Last time

Definition (Topology)

Let $\mathcal{T} \subseteq \mathcal{P}(X)$. We call $\mathcal{T}$ a *topology* on X if the following holds:

(i) $\emptyset, X \in \mathcal{T}$

(ii) Let A be an arbitrary index set. If $U_\alpha \in \mathcal{T}$ for $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$ ($\mathcal{T}$ is closed under taking arbitrary unions)

(iii) Let $n \in \mathbb{N}$. If $U_1, \ldots, U_n \in \mathcal{T}$, then $\bigcap_{i=1}^n U_i \in \mathcal{T}$ ($\mathcal{T}$ is closed under taking finite intersections)

If $U \in \mathcal{T}$, we call U *open*. We call $U \subseteq X$ *closed*, if $U^c \in \mathcal{T}$. We call $(X, \mathcal{T})$ a *topological space*.

Definition

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$ be any subset.

- The *interior* of A is $\mathring{A} := \{a \in A \colon \exists U \in \mathcal{T} \text{ s.t. } U \subseteq A \text{ and } a \in U\}$.
- The *closure* of A is $\overline{A} := \{x \in X \colon \forall U \in \mathcal{T} \text{ with } x \in U, U \cap A \neq \emptyset\}$.
- The *boundary* of A is $\partial A := \{x \in X \colon \forall U \in \mathcal{T} \text{ with } x \in U,\ U \cap A \neq \emptyset \text{ and } U \cap A^c \neq \emptyset\}$.

Density

Definition

Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is called *dense* if $\overline{A} = X$.

Using the definition of closure, we see that $A \subseteq X$ is dense if and only if for all non-empty $U \in \mathcal{T}$, $U \cap A \neq \emptyset$.

Example

- The rationals $\mathbb{Q}$ are dense in the reals $\mathbb{R}$.
- The only dense subset in $(X, \mathcal{P}(X))$ is X itself.
- Any non-empty subset is dense in $(X, \{\emptyset, X\})$.

Separability

Definition

A topological space $(X, \mathcal{T})$ is *separable* if it contains a countable dense subset.

Example

$\mathbb{R}$ is separable

$\mathbb{Q}$ is dense in $\mathbb{R}$, $\mathbb{Q}$ are countable

Hausdorff space

Definition

A topological space $(X, \mathcal{T})$ is called *Hausdorff* if for all $x \neq y \in X$ there exist open sets U_x, U_y with $x \in U_x$ and $y \in U_y$ such that $U_x \cap U_y = \emptyset$.

So in a Hausdorff space, we can separate any two elements using open sets.

Example

Let (X, d) be a metric space. Then $(X, \mathcal{T}_d)$ is Hausdorff, where $\mathcal{T}_d$ is the topology induced by the metric d.

Why?. Choose $x, y \in X$ s.t. $x \neq y$.

Let $\varepsilon = d(x,y) > 0$.

Take $U_x = B_{\varepsilon/2}(x)$ & $U_y = B_{\varepsilon/2}(y)$.

$$B_{\varepsilon/2}(x) \cap B_{\varepsilon/2}(y) = \emptyset$$

Example

Let X be an infinite set and $\mathcal{T} = \{U \subseteq X : U^c \text{ is finite}\} \cup \emptyset$. Then $(X, \mathcal{T})$ is not Hausdorff.

Proof that $(X, \mathcal{T})$ is not Hausdorff.

Suppose in order to derive a contradiction that it is Hausdorff. Take $x \neq y$, $x, y \in X$. Then $\exists\, U_x, U_y$ s.t. $x \in U_x$, $y \in U_y$, $U_x \cap U_y = \emptyset$.

Then $(U_x \cap U_y)^c = \emptyset^c = X$. By de Morgan, $(U_x \cap U_y)^c = U_x^c \cup U_y^c$, so $X = U_x^c \cup U_y^c$.

Since U_x, U_y are non-empty open sets, their complements are finite, and therefore so is their union.

Compactness

Therefore X is finite. Contradiction.

Definition

Let $(X, \mathcal{T})$ be a topological space and $K \subseteq X$.

A collection $\{U_i\}_{i \in I}$ of open sets is called *open cover* of K if $K \subseteq \cup_{i \in I} U_i$.

The set K is called *compact* if for all open covers $\{U_i\}_{i \in I}$ there exists a finite subcover, meaning there exists an $n \in \mathbb{N}$ and $\{U_1, \dots, U_n\} \subseteq \{U_i\}_{i \in I}$ such that $K \subseteq \cup_{i=1}^{n} U_i$.

Example

Let $S \subseteq X$ where $(X, \mathcal{T})$ is a topological space. If S is finite, then it is compact.

Why? Since S is finite, we can write

$$S = \{x_1, \dots, x_n\}.$$

Then for any open cover $\{U_i\}_{i \in I}$, for $j = 1, \dots, n$ $\exists\, U_j \in \{U_i\}_{i \in I}$ s.t. $x_j \in U_j$.

Then $S \subseteq \bigcup_{j=1}^{n} U_j$. $\therefore S$ is compact

Example

$(0, 1)$ is not compact.

The set $\{U_n\}_{n\in\mathbb{N}}$ where $U_n = (\frac{1}{n}, 1)$,

$(0,1) \subseteq \bigcup_{n=1}^{\infty} (\frac{1}{n}, 1)$.

Suppose that there exists a finite subcover, i.e. $N \in \mathbb{N}$ s.t. $(0,1) \subseteq \bigcup_{j=1}^{N} (\frac{1}{n_j}, 1)$

If $n \le m$, $(\frac{1}{n}, 1) \subseteq (\frac{1}{m}, 1)$ (nested sets).

$\Rightarrow (0,1) \subseteq (\frac{1}{n_N}, 1)$

Since $\exists x \in (0,1)$ s.t $0 < x < \frac{1}{n_N}$ for any n_N finite, this is a contradiction.

$\therefore \{U_n\}_{n\in\mathbb{N}}$ has no finite subcover

Proposition

Let $(X, \mathcal{T})$ be a topological space and take a non-empty subset $K \subseteq X$. The following holds:

1. If X is compact and K is closed, then K is compact (i.e. closed subsets of compact sets are compact).
2. If $(X, \mathcal{T})$ is Hausdorff, then K being compact implies that K is closed.

Proof.

(1) If X is compact and $K \subseteq X$ is closed, then K is compact

Let $\{U_i\}_{i \in I}$ be an open cover for K.

Since K is closed, K^c is open.

$\Rightarrow \{U_i\}_{i \in I} \cup K^c$ is an open cover for X

Since X is compact, there exists a finite subcover of this open cover.

The finite subcover is of the form $\{U_1, \ldots, U_n\}$ or $\{U_1, \ldots, U_n\} \cup K^c$.

Either way, $\{U_1, \ldots, U_n\}$ is a finite subcover for K (since $K \subseteq X$) $\therefore K$ is compact. $\square$

Proof.

(2) If $(X, \mathcal{T})$ is Hausdorff, then $K \subseteq X$ compact $\Longrightarrow$ K is closed.

We will show that K^c is open. We will show that there exists open sets $\{U_i\}_{i \in I}$ s.t. $K^c = \bigcup_{i \in I} U_i$.

For each $x \in K^c$, we construct an open set in K^c that contains x.

Let $x \in K^c$. Since X is Hausdorff, $\forall y \in K$, $\exists\, U_{x,y}, U_y$ disjoint, $x \in U_{x,y}$ & $y \in U_y$.

$\square$

Proof continued

Since K is compact and $\{U_y\}_{y\in K}$ is an open cover for K, $\exists y_1, \ldots, y_n$ s.t. $K \subseteq \bigcup_{i=1}^{n} U_{y_i}$.

Let $\hat{U}_x := \bigcap_{i=1}^{n} U_{x,y_i}$.

$\hat{U}_x$ is open, $x \in \hat{U}_x$, $\hat{U}_x \subseteq K^c$

$\bigcup_{x\in K^c} \hat{U}_x \subseteq K^c$; for any $x \in K^c$, $\exists \hat{U}_x$ s.t. $x \in \hat{U}_x$

$\Rightarrow K^c \subseteq \bigcup_{x\in K^c} \hat{U}_x$

$\therefore K^c = \bigcup_{x\in K^c} \hat{U}_x \Rightarrow K$ is closed

Compactness on $\mathbb{R}^n$

Theorem (Heine-Borel Theorem)

Let $K \subseteq \mathbb{R}^n$. Then K is compact with respect to the topology induced by the Euclidean distance if and only if it is closed and bounded.

Just as we had a sequential characterization of the closure of a set in metric spaces, we similarly have a sequential characterization of compactness.

Theorem

Let (X, d) be a metric space. Then $K \subset X$ is compact with respect to the metric induced by d if and only if every sequence in K admits a subsequence converging to some point in K.

A corollary of this statement together with Heine-Borel is the Bolzano-Weierstrass theorem.

Corollary (Bolzano-Weierstrass)

Any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Continuity on a topological space

Definition

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f\colon X \to Y$ is called *continuous* if for all $U \in \mathcal{T}_Y$, $f^{-1}(U) \in \mathcal{T}_X$, i.e. the pre-image of open sets is open.

We can also specify continuity at a point $x_0 \in X$.

Definition

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f\colon X \to Y$ is called *continuous at* $x_0 \in X$ if for all $U \in \mathcal{T}_Y$ with $f(x_0) \in U$, $f^{-1}(U) \in \mathcal{T}_X$, i.e. the preimage of open sets containing $f(x_0)$ is open (and contains x_0).

Proposition

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Suppose $K \subset X$ is compact and let $f : K \to Y$ be continuous. Then $f(K)$ is compact.

Recall from the set theory section:

If $f : X \to Y$:

1. $A \subseteq B \subseteq Y \Rightarrow f^{-1}(A) \subseteq f^{-1}(B)$ and $A \subseteq B \subseteq X \Rightarrow f(A) \subseteq f(B)$
2. $f^{-1}(\cup_{i \in I} A_i) = \cup_{i \in I} f^{-1}(A_i)$, where $A_i \subseteq Y \; \forall i \in I$
3. $f(\cup_{i \in I} A_i) = \cup_{i \in I} f(A_i)$, where $A_i \subseteq X \; \forall i \in I$
4. $A \subseteq X \Rightarrow A \subseteq f^{-1}(f(A))$
5. $B \subseteq Y \Rightarrow f(f^{-1}(B)) \subseteq B$

Proof.

Let $\{U_i\}_{i\in I}$ be an open cover for $f(K)$,

i.e. $f(K) \subseteq \bigcup_{i\in I} U_i$.

Then $f^{-1}(f(K)) \subseteq f^{-1}\left(\bigcup_{i\in I} U_i\right)$ by ①

$= \bigcup_{i\in I} f^{-1}(U_i)$ by ②

Since $K \subseteq f^{-1}(f(K))$.

$\therefore K \subseteq \bigcup_{i\in I} f^{-1}(U_i)$

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Proof continued

Since f is continuous, $f^{-1}(U_i)$ are open,

$\therefore$ $\{f^{-1}(U_i)\}_{i \in I}$ is an open cover for K

Since K is compact, there exist $f^{-1}(U_1), \dots, f^{-1}(U_n)$ such that $K \subseteq \bigcup_{i=1}^{n} f^{-1}(U_i)$.

$$\Rightarrow f(K) \subseteq f\left(\bigcup_{i=1}^{n} f^{-1}(U_i)\right)$$

$$= \bigcup_{i=1}^{n} f(f^{-1}(U_i))$$

$$\subseteq \bigcup_{i=1}^{n} U_i$$

$\therefore \{U_1, \ldots, U_n\}$ is a finite subcover for $f(K)$ $\therefore f(K)$ is compact

Linear Algebra

V: set, $\mathbb{F}$: field (think $\mathbb{R}$ or $\mathbb{C}$)

Definition

We call V a **vector space** if the following hold:

(A) *Commutativity in addition:* $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ for all $\mathbf{u},\mathbf{v}\in V$

(B) *Associativity in addition:* $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in V$

(C) *Existence of a neutral element, addition:* There exists a vector $\mathbf{0}$ such that for any $\mathbf{v}\in V$, $\mathbf{0}+\mathbf{v}=\mathbf{v}$

(D) *Additive inverse:* For every $\mathbf{v}\in V$, there exists another vector, which we denote $-\mathbf{v}$, such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$.

(E) *Existence of a neutral element, multiplication:* For any $\mathbf{v}\in V$, $1\times\mathbf{v}=\mathbf{v}$

(F) *Associativity in multiplication:* Let $\alpha,\beta\in\mathbb{F}$. For any $\mathbf{v}\in V$, $(\alpha\beta)\mathbf{v}=\alpha(\beta\mathbf{v})$

(G) Let $\alpha\in\mathbb{F},\mathbf{u},\mathbf{v}\in V$. $\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\overset{\alpha}{\cancel{\beta}}\mathbf{v}$.

(H) Let $\alpha,\beta\in\mathbb{F},\mathbf{v}\in V$. $(\alpha+\beta)\mathbf{v}=\alpha\mathbf{v}+\beta\mathbf{v}$.

Elements of the vector space are called vectors.
Most often we will assume $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.

Example

The following are vector spaces:

- $\mathbb{R}^n$
- $\mathbb{C}^n$
- $C(\mathbb{R};\mathbb{R})$, continuous functions from $\mathbb{R}$ to $\mathbb{R}$
- $M_{n\times m}$, matrices of size $n \times m$
- $\mathbb{P}_n$ (polynomials of degree n, $p(x) = a_0 + a_1 x + \ldots + a_n x^n$).

Lemma

For every $\mathbf{v} \in V$, $0\mathbf{v} = \mathbf{0}$.

Proof.

$$0\vec{v} = (0+0)\vec{v} = 0\vec{v} + 0\vec{v}$$

Add the additive inverse of $0\vec{v}$ to both sides:

$$\vec{0} = 0\vec{v}.$$

∎

Lemma

For every $\mathbf{v} \in V$, we have $-\mathbf{v} = (-1) \times \mathbf{v}$.

Proof.

We want to show that $\vec{v} + (-1)\vec{v} = 0$.

prove this is additive inverse

$$\vec{v} + (-1)\vec{v} = \vec{v}\,(1 + (-1)) = \vec{v}\,0 = \vec{0}$$

$\square$

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Definition

A subset U of V is called a **subspace** of of V if U is also a vector space (using the same addition and scalar multiplication as on V).

Proposition

A subset U of V is a subspace of V if and only if U satisfies the following three conditions:

1. $\mathbf{0} \in U$
2. Closed under addition: $\vec{u}, \vec{w} \in U$ implies $\mathbf{u} + \mathbf{v} \in U$
3. Closed under scalar multiplication: $\alpha \in \mathbb{F}$ and $\mathbf{u} \in U$ implies $\alpha \mathbf{u} \in U$

Proof.

($\Rightarrow$) If $\mathcal{U}$ is a subspace of V, then it satisfies these properties since it is a vector space.

($\Leftarrow$) Suppose (1), (2), & (3) hold for $\mathcal{U}$.

Take $\vec{v} \in \mathcal{U}$. Then $(-1)\vec{v} \in \mathcal{U}$, and we already proved that $(-1)\vec{v} = -\vec{v}$.

$\square$

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