Exercises for Module 8: Linear Algebra II

1. A square matrix is called *nilpotent* if $\exists k \in \mathbb{N}$ such that $A^k = 0$. Show that for a nilpotent matrix A, $|A| = 0$.

2. A real square matrix Q is called *orthogonal* if $Q^T Q = I$. Prove that if Q is orthogonal, then $|Q| = \pm 1$.

3. An $n \times n$ matrix is called *antisymmetric* if $A^T = -A$. Prove that if A is antisymmetric and n is odd, then $|A| = 0$.

4. Let V be an inner product space, U a vector space and $S\colon U \to V$, $T\colon U \to V$ be linear maps . Show that $\langle S\mathbf{u}, \mathbf{v}\rangle = \langle T\mathbf{u}, \mathbf{v}\rangle$ for all $\mathbf{u} \in U$ and $\mathbf{v} \in V$ implies $S = T$.

5. Let V be an inner product space and $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be an orthonormal basis and $\mathbf{y} \in V$. Then, $\mathbf{x}$ has a unique representation $\mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i$. Show that $\alpha_i = \langle \mathbf{y}, \mathbf{x}_i\rangle$ for all $i = 1, \ldots, n$.

6. Let V be an inner product space and $U \subseteq V$ a subset. Show that $U^{\perp}$ is a subspace of V.

7. Let U, V, W be inner product spaces and $S, T \in \mathcal{L}(U, V)$ and $R \in \mathcal{L}(V, W)$. Show that the following holds

   1. $(S + \alpha T)^* = S^* + \overline{\alpha} T^*$ for all $\alpha \in \mathbb{F}$
   2. $(S^*)^* = S$
   3. $(RS)^* = S^* R^*$
   4. $I^* = I$, where $I \colon U \to U$ is the identity operator on U