# Module 2: Set Theory

## Operational math bootcamp

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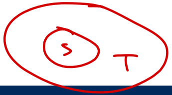
# Outline

- Review of basic set theory
- Ordered Sets
- Functions

# Introduction to Set Theory

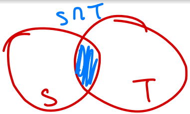
- We define a *set* to be a collection of mathematical objects.
- If $S$ is a set and $x$ is one of the objects in the set, we say $x$ is an element of $S$ and denote it by $x \in S$.
- The set of no elements is called empty set and is denoted by $\emptyset$.




## Definition (Subsets, Union, Intersection)

Let $S$, $T$ be sets.

- We say that $S$ is a *subset* of $T$, denoted $S \subseteq T$, if $s \in S$ implies $s \in T$.
- We say that $S = T$ if $S \subseteq T$ and $T \subseteq S$.
- We define the *union* of $S$ and $T$, denoted $S \cup T$, as all the elements that are in *either* $S$ or $T$.
- We define the *intersection* of $S$ and $T$, denoted $S \cap T$, as all the elements that are in *both* $S$ and $T$.
- We say that $S$ and $T$ are *disjoint* if $S \cap T = \emptyset$.



# Some examples

**Example**

$\mathbb{N} \subseteq \mathbb{N}_0 \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

**Example**

Let $a, b \in \mathbb{R}$ such that $a < b$.
Open interval: $(a, b) := \{x \in \mathbb{R} : a < x < b\}$ ($a, b$ may be $-\infty$ or $+\infty$)
Closed interval: $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$
We can also define half-open intervals.

## Example

Let $A = \{x \in \mathbb{N} : 3|x\}$ and $B = \{x \in \mathbb{N} : 6|x\}$. Show that $B \subseteq A$.

*Proof.* Let $x \in B$. Then $6|x$, so by definition $\exists k \in \mathbb{Z}$ such that $x = 6k$.

Then $x = 3 \times 2k$ so $3|x$ by definition.

Thus $x \in A$.

# Difference of sets


## Definition

Let $A, B \subseteq X$. We define the *set-theoretic difference* of $A$ and $B$, denoted $A \setminus B$ (sometimes $A - B$) as the elements of $X$ that are in $A$ but *not* in $B$.
The complement of a set $A \subseteq X$ is the set $A^c := X \setminus A$.


## Example

Let $X \subseteq \mathbb{R}$ be defined as $X = \{x \in \mathbb{R} : 0 < x \leq 40\} = (0, 40]$. Then

$X^c = \{x \in \mathbb{R} : x \leq 0 \text{ or } x > 40\} = (-\infty, 0] \cup (40, \infty)$

Recall that for sets $S$, $T$:

- the *union* of $S$ and $T$, denoted $S \cup T$, is all the elements that are in *either* $S$ and $T$
- and the *intersection* of $S$ and $T$, denoted $S \cap T$, is all the elements that are in *both* $S$ and $T$.

We extend the definition of union and intersection to an arbitrary family of sets as follows:

## Definition

Let $S_\alpha$, $\alpha \in A$, be a family of sets. $A$ is called the *index set*. We define

$$\bigcup_{\alpha \in A} S_\alpha := \{x : \exists \alpha \text{ such that } x \in S_\alpha\},$$

$$\bigcap_{\alpha \in A} S_\alpha := \{x : x \in S_\alpha \text{ for all } \alpha \in A\}.$$

### Example

$$\bigcup_{n=1}^{\infty} [-n, n] = \mathbb{R}$$

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

## Theorem (De Morgan's Laws)

*Let $\{S_\alpha\}_{\alpha \in A}$ be an arbitrary collection of sets. Then*

$$\underbrace{\left(\bigcup_{\alpha\in A} S_\alpha\right)^c = \bigcap_{\alpha\in A} S_\alpha^c}_{1} \quad \text{and} \quad \left(\bigcap_{\alpha\in A} S_\alpha\right)^c = \bigcup_{\alpha\in A} S_\alpha^c$$

*Proof.*

Let $x \in \left(\bigcup_{\alpha\in A} S_\alpha\right)^c$. This is true if and only if

$$x \notin \bigcup_{\alpha\in A} S_\alpha \iff x \in S_\alpha^c \ \forall \alpha \in A$$

$$\iff x \in \bigcap_{\alpha\in A} S_\alpha^c$$

2 exercise

Since a set is itself a mathematical object, a set can itself contain sets.

## Definition

The power set $\mathcal{P}(S)$ of a set $S$ is the set of all subsets of $S$.

## Example

Let $S = \{a, b, c\}$.
Then $\mathcal{P}(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \underbrace{\{a,b,c\}}_{=S}\}$

$2^n$

Another way of building a new set from two old ones is the Cartesian product of two sets.

**Definition**

Let $S, T$ be sets. The *Cartesian product* $S \times T$ is defined as the set of tuples with elements from $S, T$, i.e

$$S \times T = \{(s,t) : \; s \in S \; \text{ and } \; t \in T\}.$$

This can also be extended inductively to a finite family of sets.

# Ordered set

$(x_1, x_2)$

**Definition**

A *relation* $R$ on a set $X$ is a subset of $X \times X$. A relation $\leq$ is called a *partial order* on $X$ if it satisfies

1. reflexivity: $x \leq x \quad \forall x \in X$
2. transitivity: $x, y, z \in X, \; x \leq y, \; y \leq z \Rightarrow x \leq z$
3. anti-symmetry: $x, y \in X, \; x \leq y$ and $y \leq x$ implies $x = y$

The pair $(X, \leq)$ is called a *partially ordered set*.

A *chain* or *totally ordered set* $C \subseteq X$ is a subset with the property $x \leq y$ or $y \leq x$ for any $x, y \in C$.

## Example

The real numbers with the usual ordering, $(\mathbb{R}, \leq)$ are totally ordered.

## Example

The power set of a set $X$ with the ordering given by $\subseteq$, $(\mathcal{P}(X), \subseteq)$ is a partially ordered set. $\not\leq$

## Example

Let $X = \{a, b, c, d\}$. What is $\mathcal{P}(X)$? Find a chain in $\mathcal{P}(X)$.

$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{c, d\}, \{b, d\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{b, c, d\}, \{a, b, d\}, \{a, c, d\}, X\}$

$$C = \{\phi, \{a\}, \{a,b\}, \{a,b,c\}, X\}$$

## Example

Consider the set $C([0,1],\mathbb{R}) := \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$.

For two functions $f, g \in C([0,1],\mathbb{R})$, we define the ordering as $f \leq g$ if $f(x) \leq g(x)$ for $x \in [0,1]$. Then $(C([0,1],\mathbb{R}), \leq)$ is a partially ordered set.

Can you think of a chain that is a subset of $(C([0,1],\mathbb{R})$?

$\hookrightarrow$ exercise

## Definition

A non-empty partially ordered set $(X, \leq)$ is *well-ordered* if every non-empty subset $A \subseteq X$ has a mimimum element.

Example:

$(\mathbb{N}, \leq)$ is... well-ordered

$(\mathbb{R}, \leq)$ is... not well-ordered

$S = (0, 1)$

$(\mathbb{R}, \leq)$

## Definition

Let $(X, \leq)$ be a partially ordered set and $S \subseteq X$.

Then $x \in X$ is an *upper bound* for $S$ if for all $s \in S$ we have $s \leq x$.
Similarly, $y \in X$ is a *lower bound* for $S$ if for all $s \in S$, $y \leq s$.

If there exists an upper bound for $S$, we call $S$ *bounded above* and if there exists a lower bound for $S$, we call $S$ *bounded below*. If $S$ is bounded above and bounded below, we say $S$ is *bounded*.

We can also ask if there exists a least upper bound or a greatest lower bound.

## Definition

Let $(X, \leq)$ be a partially ordered set and $S \subseteq X$.

We call $x \in X$ *least upper bound* or *supremum*, denoted $x = \sup S$, if $x$ is an upper bound and for any other upper bound $y \in X$ of $S$ we have $x \leq y$.

Likewise, $x \in X$ is the *greatest lower bound* or *infimum* for $S$, denoted $x = \inf S$, if it is a lower bound and for any other lower bound $y \in X$, $y \leq x$.

Note that the supremum and infimum of a bounded set do not necessarily need to exist. However, if they do exists they are unique, which justifies the article *the* (exercise). Nevertheless, the reals have a remarkable property, which we will take as an axiom.

### Completeness Axiom

Let $S \subseteq \mathbb{R}$ be bounded above. Then there exists $r \in \mathbb{R}$ such that $r = \sup S$, i.e. $S$ has a least upper bound.

By setting $S' = -S := \{-s : s \in S\}$ and noting $\inf S = -\sup S'$, we obtain a similar statement for infima if $S$ is bounded below. As mentioned above, this property is fairly special, for example it fails for the rationals.

### Example

Let $S = \{q \in \mathbb{Q} : x^2 < 7\}$. Then $S$ is bounded above in $\mathbb{Q}$, but there exists no least upper bound in $\mathbb{Q}$.

There is a nice alternative characterization for suprema in the real numbers.

$A \Rightarrow B$
$\neg B \Rightarrow \neg A$

## Proposition

*Let $S \subseteq \mathbb{R}$ be bounded above. Then $r = \sup S$ if and only if $r$ is an upper bound and for all $\epsilon > 0$ there exists an $s \in S$ such that $r - \epsilon < s$.*

*Proof.* ($\Rightarrow$) We prove the contrapositive.
Suppose $r$ is either not an upper bound or $\exists \varepsilon > 0$ such that $\forall s \in S \;\; r - \varepsilon \geq s$.
If $r$ is not an upper bound for $S$, then $r \neq \sup S$.
In the second case, $r - \varepsilon$ is an upper bound for $S$. Since $\varepsilon > 0$, $r - \varepsilon < r$, so $r \neq \sup S$.

*Proof.* ($\Leftarrow$) By contradiction.

Suppose $r$ is an upper bound for $S$ and $\underbrace{\forall \varepsilon > 0 \ \exists s \in S \text{ such that } r - \varepsilon < s}$, but $r \neq \sup S$

$\Rightarrow r > \sup S \Rightarrow \underline{r - \sup S} > 0$

By assumption (use $\varepsilon = r - \sup S$), $\exists s \in S$ such that $r - (r - \sup S) < s$.

$\Rightarrow \sup S < s$. $\Rightarrow\!\Leftarrow$

$\therefore r = \sup S$

Using the same trick, we may obtain a similar result for infima.

## Example

Consider $S = \{1/n : n \in \mathbb{N}\}$. Then $\sup S = 1$ and $\underbrace{\inf S = 0}$.

# Functions

**Definition**

A function $f$ from a set $X$ to a set $Y$ is a subset of $X \times Y$ with the properties:

1. For every $x \in X$, there exists a $y \in Y$ such that $(x, y) \in f$
2. If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

$X$ is called the *domain* of $f$.

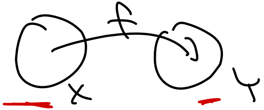
How does this connect to other descriptions of functions you may have seen?

$f : X \to Y \qquad x \mapsto y$ where $(x, y) \in f$

**Example**

For a set $X$, the identity function is:

$$1_X : X \to X, \quad x \mapsto x$$




## Definition (Image and pre-image)

Let $f : X \to Y$ and $A \subseteq X$ and $B \subseteq Y$.

- The *image* of $f$ is the set $f(A) := \{f(x) : x \in A\}$.
- The *pre-image* of $f$ is the set $f^{-1}(B) := \{x : f(x) \in B\}$.

Helpful way to think about it for proofs:

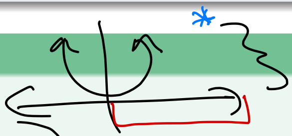
**Image:** If $y \in f(A)$, then $y \in Y$, and there exists an $x \in A$ such that $y = f(x)$.

**Pre-image:** If $x \in f^{-1}(B)$, then $x \in X$ and $f(x) \in B$.

## Definition (Surjective, injective and bijective)

Let $f : X \to Y$, where $X$ and $Y$ are sets. Then

- $f$ is *injective* if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$
- $f$ is *surjective* if for every $y \in Y$, there exists an $x \in X$ such that $y = f(x)$
- $f$ is *bijective* if it is both injective and ~~bijective~~ surjective




## Example

Let $f : X \to Y$, $x \mapsto x^2$.

$f$ is surjective if $X = \mathbb{R}$, $Y = [0, \infty)$

$f$ is injective if $X = \mathbb{N}$, $Y = \mathbb{R}$

$[0, \infty)$

$f$ is bijective if $X = Y = [0, \infty)$

$f$ is neither surjective nor injective if

$\hookrightarrow X = Y = \mathbb{R}$

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