

## Exercises for Module 3: Set Theory II and Metric Spaces I

1. Show that for sets  $A, B \subseteq X$  and  $f : X \rightarrow Y$ ,  $f(A \cap B) \subseteq f(A) \cap f(B)$ .

2. Let  $f : X \rightarrow Y$  and  $B \subseteq Y$ . Prove that  $f(f^{-1}(B)) \subseteq B$ , with equality iff  $f$  is surjective.

3. Prove that  $f(\cup_{i \in I} A_i) = \cup_{i \in I} f(A_i)$ , where  $f : X \rightarrow Y$ ,  $A_i \subseteq X \forall i \in I$ .

4. Show that  $\mathbb{N}$  and  $\mathbb{Z}$  have the same cardinality.

5. Show that  $|(0, 1)| = |(1, \infty)|$ .

6. Show that the infinity norm  $\|x\|_\infty$ ,  $x \in \mathbb{R}^n$ , is a norm.

7. Let  $(X, d)$  be any metric space, and define  $\tilde{d} : X \times X \rightarrow \mathbb{R}$  by

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \quad x, y \in X.$$

Show that  $\tilde{d}$  is a metric on  $X$ .

8. Let  $X$  be a set and define  $d : X \times X \rightarrow \mathbb{R}$  by  $d(x, x) = 0$  and  $d(x, y) = 1$  for  $x \neq y \in X$ . Prove that  $d$  is a metric on  $X$ . What do open balls look like for different radii  $r > 0$ ?