Module 3: Set theory and metrics

Operational math bootcamp

Statistical Sciences
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Outline

- More on set theory
- Cardinality of sets
- Metrics and norms

Recall

Definition (Image and pre-image)

Let $f : X \to Y$ and $A \subseteq X$ and $B \subseteq Y$.

- The *image* of f is the set $f(A) := \{f(x) : x \in A\}$.
- The *pre-image* of f is the set $f^{-1}(B) := \{x : f(x) \in B\}$.

Definition (Surjective, injective and bijective)

Let $f : X \to Y$, where X and Y are sets. Then

- f is *injective* if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$
- f is *surjective* if for every $y \in Y$, there exists an $x \in X$ such that $y = f(x)$
- f is *bijective* if it is both injective and surjective

Proposition

Let $f : X \to Y$ and $A \subseteq X$. Prove that $A \subseteq f^{-1}(f(A))$, with equality iff f is injective.

Proof.

Cardinality

Intuitively, the *cardinality* of a set A, denoted $|A|$, is the number of elements in the set. For sets with only a finite number of elements, this intuition is correct. We call a set with finitely many elements finite.

We say that the empty set has cardinality 0 and is finite.

Proposition

If X is finite set of cardinality n, then the cardinality of $\mathcal{P}(X)$ is 2^n.

Proof.

Definition

Two sets A and B have same cardinality, $|A| = |B|$, if there exists bijection $f : A \to B$.

Example

Which is bigger, $\mathbb{N}$ or $\mathbb{N}_0$?

Cantor-Schröder-Bernstein

Definition

We say that the cardinality of a set A is less than the cardinality of a set B, denoted $|A| \leq |B|$ if there exists an injection $f : A \to B$.

Theorem (Cantor-Bernstein)

Let A, B, be sets. If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Example

$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$

Proof that $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$:

Definition

Let A be a set.

1. A is *finite* if there exists an $n \in \mathbb{N}$ and a bijection $f : \{1, \dots, n\} \to A$
2. A is *countably infinite* if there exists a bijection $f : \mathbb{N} \to A$
3. A is *countable* if it is finite or countably infinite
4. A is *uncountable* otherwise

Example

The rational numbers are countable, and in fact $|\mathbb{Q}| = |\mathbb{N}|$.

Proof. First we show $|\mathbb{N}| \leq |\mathbb{Q}^+|$.

Next, we show that $|\mathbb{Q}^+| \leq |\mathbb{N} \times \mathbb{N}|$.

We can extend this to $\mathbb{Q}$ as follows:

Theorem

The cardinality of $\mathbb{N}$ is smaller than that of $(0, 1)$.

Proof.

First, we show that there is an injective map from $\mathbb{N}$ to $(0, 1)$.

Next, we show that there is no surjective map from $\mathbb{N}$ to (0, 1). We use the fact that every number $r \in (0, 1)$ has a binary expansion of the form $r = 0.\sigma_1\sigma_2\sigma_3\ldots$ where $\sigma_i \in \{0, 1\}$, $i \in \mathbb{N}$. □

Proof.

Now we suppose in order to derive a contradiction that there does exist a surjective map f from $\mathbb{N}$ to (0, 1)., i.e. for $n \in \mathbb{N}$ we have $f(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\ldots$. This means we can list out the binary expansions, for example like

$$
\begin{aligned}
f(1) =& 0.00000000\ldots \\
f(2) =& 0.1111111111\ldots \\
f(3) =& 0.0101010101\ldots \\
f(4) =& 0.1010101010\ldots
\end{aligned}
$$

We will construct a number $\tilde{r} \in (0, 1)$ that is not in the image of f. □

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Proof.

Define $\tilde{r} = 0.\tilde{\sigma}_1\tilde{\sigma}_2\ldots$, where we define the nth entry of $\tilde{r}$ to be the the opposite of the nth entry of the nth item in our list:

$$\tilde{\sigma}_n = \begin{cases} 1 & \text{if } \sigma_n(n) = 0, \\ 0 & \text{if } \sigma_n(n) = 1. \end{cases}$$

Then $\tilde{r}$ differs from $f(n)$ at least in the nth digit of its binary expansion for all $n \in \mathbb{N}$. Hence, $\tilde{r} \notin f(\mathbb{N})$, which is a contradiction to f being surjective. This technique is often referred to as Cantor's diagonal argument. □

Proposition

(0,1) and $\mathbb{R}$ have the same cardinality.

Proof.

□

We have shown that there are different sizes of infinity, as the cardinality of $\mathbb{N}$ is infinite but still smaller than that of $\mathbb{R}$ or $(0,1)$. In fact, we have

$$|\mathbb{N}| \quad |\mathbb{N}_0| \quad |\mathbb{Z}| \quad |\mathbb{Q}| \quad |\mathbb{R}|.$$

Because of this, there are special symbols for these two cardinalities: The cardinality of $\mathbb{N}$ is denoted $\aleph_0$, while the cardinality of $\mathbb{R}$ is denoted $\mathfrak{c}$.

In fact there are many other cardinalities, as the following theorem shows:

Theorem (Cantor's theorem)

For any set A, $|A| < |\mathcal{P}(A)|$.

Metric Spaces

Definition (Metric)

A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ that satisfies:

(a) Positive definiteness:

(b) Symmetry:

(c) Triangle inequality:

A set together with a metric is called a metric space.

Example ($\mathbb{R}^n$ with the Euclidean distance)

Definition (Norm)

A *norm* on an $\mathbb{F}$-vector space E is a function $\|\cdot\| : E \to \mathbb{R}$ that satisfies:

(a) Positive definiteness:

(b) Homogeneity:

(c) Triangle inequality:

A vector space with a norm is called a normed space. A normed space is a metric space using the metric $d(x, y) = \|x - y\|$.

Example (p-norm on $\mathbb{R}^n$)

The p-norm is defined for $p \geq 1$ for a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ as

The infinity norm is the limit of the p-norm as $p \to \infty$, defined as

Example (p-norm on $C([0,1];\mathbb{R})$)

If we look at the space of continuous functions $C([0,1];\mathbb{R})$, the p-norm is

and the $\infty-$norm (or sup norm) is

Definition

A subset A of a metric space (X, d) is *bounded* if there exists $M > 0$ such that $d(x, y) < M$ for all $x, y \in A$.

Definition

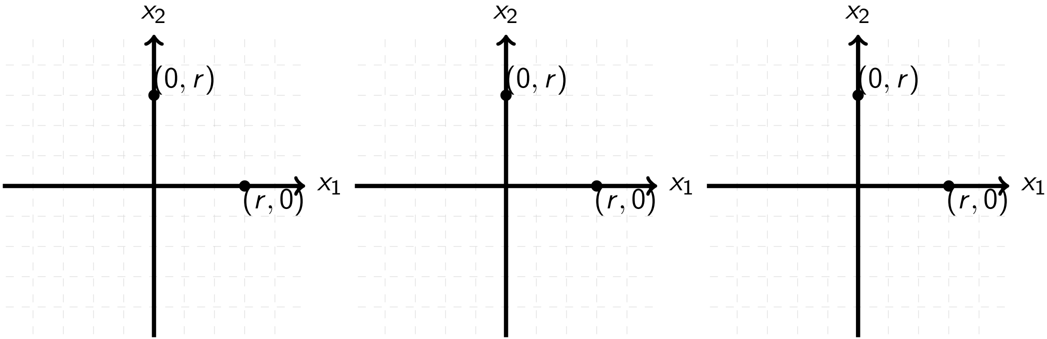
Let (X, d) be a metric space. We define the *open ball* centred at a point $x_0 \in X$ of radius $r > 0$ as

$$B_r(x_0) := \{x \in X : d(x, x_0) < r\}.$$

Example

In $\mathbb{R}$ with the usual norm (absolute value), open balls are symmetric open intervals, i.e.

Example: Open ball in $\mathbb{R}^2$ with different metrics



(a) 1-norm (taxicab metric) (b) 2-norm (Euclidean metric) (c) ∞-norm

Figure: $B_r(0)$ for different metrics

References

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