Module 3: Set theory and metrics

Operational math bootcamp

Statistical Sciences
UNIVERSITY OF TORONTO

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July 13, 2023

Outline

- More on set theory
- Cardinality of sets
- Metrics and norms

Recall

Definition (Image and pre-image)

Let $f : X \to Y$ and $A \subseteq X$ and $B \subseteq Y$.

- The *image* of f is the set $f(A) := \{f(x) : x \in A\}$.
- The *pre-image* of f is the set $f^{-1}(B) := \{x : f(x) \in B\}$.

Definition (Surjective, injective and bijective)

Let $f : X \to Y$, where X and Y are sets. Then

- f is *injective* if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$
- f is *surjective* if for every $y \in Y$, there exists an $x \in X$ such that $y = f(x)$
- f is *bijective* if it is both injective and bijective

Proposition

Let $f : X \to Y$ and $A \subseteq X$. Prove that $A \subseteq f^{-1}(f(A))$, with equality iff f is injective.

Proof. First we show $A \subseteq f^{-1}(f(A))$.

Let $x \in A$. Then $f(x) \in f(A)$ by definition. Again by definition, $x \in f^{-1}(f(A))$.

Now suppose f is injective. We want to show that $f^{-1}(f(A)) \subseteq A$. Let $x \in f^{-1}(f(A))$. By def, $f(x) \in f(A)$. By definition, $\exists \tilde{x} \in A$ such that $f(x) = f(\tilde{x})$. Since f is injective, $x = \tilde{x}$. $\therefore x \in A$.

Cardinality

Intuitively, the *cardinality* of a set A, denoted $|A|$, is the number of elements in the set. For sets with only a finite number of elements, this intuition is correct. We call a set with finitely many elements finite.

We say that the empty set has cardinality 0 and is finite.

Proposition

(*) *If X is finite set of cardinality n, then the cardinality of $\mathcal{P}(X)$ is 2^n.*

Proof. We prove this by induction on $n \in \mathbb{N}_0$.

<u>Base case</u>: $n = 0 \implies X = \emptyset$

$\mathcal{P}(X) = \mathcal{P}(\emptyset) = \{\emptyset\}$. So $\mathcal{P}(X)$ has cardinality $1 = 2^0$. The statement holds for $n = 0$.

<u>Inductive hypothesis</u>: suppose (*) holds for some $n \in \mathbb{N}_0$.

Let X be a set with cardinality $n+1$, i.e. X has $n+1$ elements. Let's call them $x_1, \dots, x_{n+1}$.

$$\begin{aligned} X &= \{x_1, x_2, \dots, x_n, x_{n+1}\} \\ &= \underbrace{\{x_1, x_2, \dots, x_n\}}_{A} \cup \{\underbrace{x_{n+1}}_{B}\} \end{aligned}$$

$\Rightarrow \mathcal{P}(A) = 2^n$ by IH

Any subset of X must either be a subset of A or contain x_{n+1}. Let's count those of the latter form.

$$1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$\{x_{n+1}\}$ $\{x_i, x_{n+1}\}$
$i=1,\ldots,n$

Therefore $|\mathcal{P}(X)| = 2^n + 2^n = 2^{n+1}$!

Thus the claim holds by induction.

Definition

Two sets A and B have same cardinality, $|A| = |B|$, if there exists bijection $f : A \to B$.

Example

$\{1, 2, 3, \ldots\}$ $\{0, 1, 2, 3, \ldots\}$

Which is bigger, $\mathbb{N}$ or $\mathbb{N}_0$?

$$|\mathbb{N}| = |\mathbb{N}_0|$$

$f : \mathbb{N}_0 \to \mathbb{N}$ $\quad n \mapsto n+1$ is a bijection

Cantor-Schröder-Bernstein

Definition

We say that the cardinality of a set A is less than the cardinality of a set B, denoted $|A| \leq |B|$ if there exists an injection $f : A \to B$.

Theorem (Cantor-Bernstein)

Let A, B, be sets. If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Example

$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$

Proof that $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$: First we show $|\mathbb{N}| \le |\mathbb{N} \times \mathbb{N}|$.

$f: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ $n \mapsto (n, 1)$ is an injection, so $|\mathbb{N}| \le |\mathbb{N} \times \mathbb{N}|$.

Next, show $|\mathbb{N} \times \mathbb{N}| \le |\mathbb{N}|$.

$g: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ $(n, m) \mapsto \underbrace{2^n 3^m}_{\in \mathbb{N}}$

We claim g is an injection. Let $n_1, n_2, m_1, m_2 \in \mathbb{N}$ such that $2^{n_1} 3^{m_1} = 2^{n_2} 3^{m_2}$. By the Fundamental Thm of Arithmetic, we must have $n_1 = n_2$, $m_1 = m_2$.

$\therefore g$ is injective, $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$ by CB Thm

Definition

Let A be a set.

1. A is *finite* if there exists an $n \in \mathbb{N}$ and a bijection $f : \{1, \ldots, n\} \to A$
2. A is *countably infinite* if there exists a bijection $f : \mathbb{N} \to A$
3. A is *countable* if it is finite or countably infinite
4. A is *uncountable* otherwise

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Example

The rational numbers are countable, and in fact $|\mathbb{Q}| = |\mathbb{N}|$.

Proof. First we show $|\mathbb{N}| \leq |\mathbb{Q}^+|$. $\quad \mathbb{Q}^+ := \{x \in \mathbb{Q} : x > 0\}$

$$
\begin{array}{ccccc}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
2 & \cancel{\frac{2}{2}} & \frac{2}{3} & \cancel{\frac{2}{4}} & \cdots \\
3 & \frac{3}{2} & \cancel{\frac{3}{3}} & \frac{3}{4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

This listing method is an injection from $\mathbb{N}$ to $\mathbb{Q}^+$.

Next, we show that $|\mathbb{Q}^+| \leq |\mathbb{N} \times \mathbb{N}|$.

$f: \mathbb{Q}^+ \to \mathbb{N} \times \mathbb{N} \; : \; \frac{p}{q} \to (p, q)$

is an injection.

Since $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$, then $\underline{|\mathbb{N}| = |\mathbb{Q}^+|}$.

We can extend this to $\mathbb{Q}$ as follows:

Let $f : \mathbb{N} \to \mathbb{Q}^+$ be a bijection.

Then define $g : \mathbb{N} \to \mathbb{Q}$ as follows:

$$g(1) = 0$$

$$g(n) = \begin{cases} f(n) & n \text{ is even} \\ -f(n) & n \text{ is odd} \end{cases}$$

for $n > 1$

g is bijection $\therefore$ $\mathbb{Q}$ is countable

Theorem

The cardinality of $\mathbb{N}$ is smaller than that of $(0, 1)$.

Proof.

First, we show that there is an injective map from $\mathbb{N}$ to $(0, 1)$.

$$f : \mathbb{N} \to (0,1) \qquad n \mapsto \frac{1}{n}$$

Next, we show that there is no surjective map from $\mathbb{N}$ to (0, 1). We use the fact that every number $r \in (0, 1)$ has a binary expansion of the form $r = 0.\sigma_1\sigma_2\sigma_3\ldots$ where $\sigma_i \in \{0, 1\}$, $i \in \mathbb{N}$. □

Proof.

Now we suppose in order to derive a contradiction that there does exist a surjective map f from $\mathbb{N}$ to $(0, 1)$., i.e. for $n \in \mathbb{N}$ we have $f(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\ldots$. This means we can list out the binary expansions, for example like

$$
\begin{aligned}
f(1) &= 0.00000000\ldots \\
f(2) &= 0.1111111111\ldots \\
f(3) &= 0.0101010101\ldots \\
f(4) &= 0.1010101010\ldots
\end{aligned}
$$

We will construct a number $\tilde{r} \in (0, 1)$ that is not in the image of f. □

Proof.

Define $\tilde{r} = 0.\tilde{\sigma}_1\tilde{\sigma}_2\ldots$, where we define the nth entry of $\tilde{r}$ to be the the opposite of the nth entry of the nth item in our list:

$$\tilde{\sigma}_n = \begin{cases} 1 & \text{if } \sigma_n(n) = 0, \\ 0 & \text{if } \sigma_n(n) = 1. \end{cases}$$

Then $\tilde{r}$ differs from $f(n)$ at least in the nth digit of its binary expansion for all $n \in \mathbb{N}$. Hence, $\tilde{r} \notin f(\mathbb{N})$, which is a contradiction to f being surjective. This technique is often referred to as Cantor's diagonal argument. $\square$

Proposition

(0,1) and $\mathbb{R}$ have the same cardinality.

Proof.

$f : \mathbb{R} \to (0,1) \quad x \mapsto \frac{1}{\pi}\left(\arctan(x) + \frac{\pi}{2}\right)$

is a bijection

□

We have shown that there are different sizes of infinity, as the cardinality of $\mathbb{N}$ is infinite but still smaller than that of $\mathbb{R}$ or $(0, 1)$. In fact, we have

$$|\mathbb{N}| = |\mathbb{N}_0| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|.$$

Because of this, there are special symbols for these two cardinalities: The cardinality of $\mathbb{N}$ is denoted $\aleph_0$, while the cardinality of $\mathbb{R}$ is denoted $\mathfrak{c}$.

Metric Spaces

Definition (Metric)

A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ that satisfies:

(a) Positive definiteness: $d(x,y) \geq 0 \;\; \forall x,y \in X$ and
$d(x,y) = 0 \iff x = y$

(b) Symmetry: $x,y \in X$, then $d(x,y) = d(y,x)$

(c) Triangle inequality: $x,y,z \in X \; : \; d(x,y) + d(y,z) \geq d(x,z)$

A set together with a metric is called a metric space.

Example ($\mathbb{R}^n$ with the Euclidean distance)

$$d(x,y) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2} \qquad x, y \in \mathbb{R}^n$$

$\mathbb{R}^n$ with the Euclidean distance is a metric space

$|x-y|$ absolute value on $\mathbb{R}^1$

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Definition (Norm)

A *norm* on an $\mathbb{F}$-vector space E is a function $\|\cdot\| : E \to \mathbb{R}$ that satisfies:

(a) Positive definiteness: $\|x\| \geq 0 \ \forall x \in E$ and $\|x\| = 0 \iff x = 0$

(b) Homogeneity: $x \in E$, $\alpha \in \mathbb{F}$, $\|\alpha x\| = |\alpha| \, \|x\|$

(c) Triangle inequality: $x, y \in E$ $\|x + y\| \leq \|x\| + \|y\|$

A vector space with a norm is called a normed space. A normed space is a metric space using the metric $d(x, y) = \|x - y\|$.

Example (p-norm on $\mathbb{R}^n$)

The p-norm is defined for $p \geq 1$ for a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ as

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

The infinity norm is the limit of the p-norm as $p \to \infty$, defined as

$$\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$$

Example (p-norm on $C([0,1];\mathbb{R})$)

If we look at the space of continuous functions $C([0,1];\mathbb{R})$, the p-norm is

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p}$$

and the ∞−norm (or sup norm) is

$$\|f\|_\infty = \max_{x \in [0,1]} |f(x)|$$

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Definition

A subset A of a metric space (X, d) is *bounded* if there exists $M > 0$ such that $d(x, y) < M$ for all $x, y \in A$.

Definition

Let (X, d) be a metric space. We define the *open ball* centred at a point $x_0 \in X$ of radius $r > 0$ as

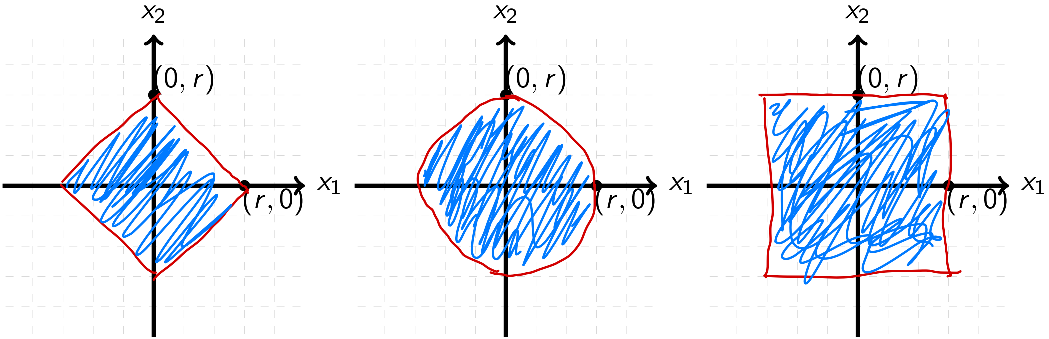
$$B_r(x_0) := \{x \in X : d(x, x_0) < r\}.$$

Example

In $\mathbb{R}$ with the usual norm (absolute value), open balls are symmetric open intervals, i.e.

$$B_r(x_0) = (x_0 - r, x_0 + r)$$

Example: Open ball in $\mathbb{R}^2$ with different metrics



(a) 1-norm (taxicab metric) (b) 2-norm (Euclidean metric) (c) ∞-norm

$d(x,y) = \sum_{i=1}^{2} |x_i - y_i|$ Figure: $B_r(0)$ for different metrics $\max_{j=1,2} |x_j - y_j|$

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