Module 4: Metric Spaces and Sequences II

1. Show that the infinite intersection of open sets may not be open and that the infinite union of closed sets may not be closed.

Consider subsets of $\mathbb{R}$.

Let $S_n = (-\frac{1}{n}, \frac{1}{n})$ for $n \in \mathbb{N}$. S_n is open for every $n \in \mathbb{N}$ but $\bigcap_{n=1}^{\infty} S_n = \{0\}$ which is closed (since $(-\infty, 0) \cup (0, \infty)$ is open).

Let $E_n = [\frac{1}{n}, 1]$ for $n \in \mathbb{N}$.
Then $\bigcup_{n=1}^{\infty} E_n = (0, 1]$ which is not closed.

2. Find the closure, interior, and boundary of the following sets using Euclidean distance:

(i) $\underbrace{\{(x, y) \in \mathbb{R}^2 : y < x^2\}}_{:= A} \subseteq \mathbb{R}^2$

(ii) $\underbrace{[0, 1) \times [0, 1)}_{:= B} \subseteq \mathbb{R}^2$

(iii) $\underbrace{\{0\} \cup \{1/n : n \in \mathbb{N}\}}_{:= C} \subseteq \mathbb{R}$

(i) $\overline{A} = \{(x, y) \in \mathbb{R}^2 : y \leq x^2\}$

$\mathring{A} = A$

$\partial A = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$

(ii) $\overline{B} = [0, 1] \times [0, 1]$

$\mathring{B} = (0, 1) \times (0, 1)$

$\partial B = \{0\} \times [0, 1] \cup \{1\} \times [0, 1] \cup [0, 1] \times \{0\} \cup [0, 1] \times \{1\}$

(iii) $\overline{C} = C$

$\mathring{C} = \emptyset$

$\partial C = C$

3. Prove the following: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X,d) that converges to a point $x \in X$. Then x is unique.

Proof. By contradiction.

Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in a metric space (X,d) that converges to both $x_1 \in X$ & $x_2 \in X$, where $x_1 \neq x_2$.

Note that since $x_1 \neq x_2$, by property (i) of metrics, $\exists\, \delta > 0$ s.t $d(x_1, x_2) = \delta$ (i.e. it is non-zero).

Let $\varepsilon > 0$ be arbitrary.

Since $x_n \to x_1$, $\exists\, n_1 \in \mathbb{N}$ s.t. $d(x_n, x_1) < \varepsilon/2$ $\forall n \geq n_1$.

Similarly, since $x_n \to x_2$, $\exists\, n_2 \in \mathbb{N}$ s.t. $d(x_n, x_2) < \varepsilon/2$ $\forall n \geq n_2$.

Let $n \geq \max\{n_1, n_2\}$. Then by the Δ inequality,

$$d(x_1, x_2) \leq d(x_1, x_n) + d(x_n, x_2) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this holds for $\varepsilon = \delta$, which is a contradiction. $\therefore x_1 = x_2$.

4. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$ such that $x_n \to x$ and and $y_n \to y$, with $\alpha, x, y, \in \mathbb{R}$.

(i) Show that $\alpha\, x_n \to \alpha\, x$.

(i) Show that $x_n + y_n \to x + y$.

(i) Let $x_n \to x$.

Let $\varepsilon > 0$. Since $x_n \to x$, $\exists\, n_0 \in \mathbb{N}$ s.t $\forall n \geq n_0$, $|x_n - x| < \frac{\varepsilon}{|\alpha|}$.

This implies $\exists\, n_0 \in \mathbb{N}$ s.t $|\alpha||x_n - x| < \varepsilon$

$\Rightarrow$ $\exists\, n_0 \in \mathbb{N}$ s.t $|\alpha x_n - \alpha x| < \varepsilon$.

$\therefore \alpha x_n \to \alpha x$

(ii) Let $\varepsilon > 0$ arbitrary.

Since $x_n \to x$, $\exists\, n_x \in \mathbb{N}$ s.t. $\forall n \geq n_x$, $|x_n - x| < \varepsilon/2$.

Since $y_n \to y$, $\exists\, n_y \in \mathbb{N}$ s.t. $\forall n \geq n_y$, $|y_n - y| < \varepsilon/2$.

Let $n^* = \max\{n_x, n_y\}$.

Then for $n \geq n^*$,
$$\begin{aligned} |x_n + y_n - (x+y)| &= |x_n - x + y_n - y| \\ &\leq |x_n - x| + |y_n - y| \quad \text{by } \Delta \text{ ineq} \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

$\therefore x_n + y_n \to x + y$.

5. Show that discrete metric spaces (i.e. those with the metric defined as define $d\colon X \times X \to \mathbb{R}$ by $d(x,x) = 0$ and $d(x,y) = 1$ for $x \neq y \in X$) are complete.

Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (X,d).

Then $\forall \varepsilon > 0\ \exists n_\varepsilon \in \mathbb{N}$ s.t. $d(x_n, x_m) < \varepsilon\ \forall n,m \geq n_\varepsilon$.

Since this holds for $\varepsilon = 1$, we must have that $\exists n_1 \in \mathbb{N}$ s.t. $x_n = x_m\ \forall n,m \geq n_1$.

Therefore every Cauchy sequence in (X,d) is eventually constant, so every Cauchy sequence converges.

$\therefore (X,d)$ is complete