# Module 4: Metric Spaces II

## Operational math bootcamp

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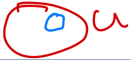
# Outline

- Open and closed sets
- Sequences
  - Cauchy sequences
  - subsequences
- Continuous functions



## Definition (Open and closed sets)

Let $(X, d)$ be a metric space.

- A set $U \subseteq X$ is *open* if for every $x \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$.
- A set $F \subseteq X$ is *closed* if $F^c := X \setminus F$ is open.

**Note:** $\emptyset$, $X$ are both open and closed

## Proposition

*Let $(X, d)$ be a metric space.*

1. *Let $A_1, A_2 \subseteq X$. If $A_1$ and $A_2$ are open, then $A_1 \cap A_2$ is open.*
2. *If $A_i \subseteq X$, $i \in I$ are open, then $\cup_{i \in I} A_i$ is open.*

*Proof.* (1) Let $A_1, A_2 \subseteq X$. If $A_1$ and $A_2$ are open, then $A_1 \cap A_2$ is open.

Let $x \in (A_1 \cap A_2)$. Then $x \in A_1$, $\exists \varepsilon_1 > 0$ s.t. $B_{\varepsilon_1}(x) \subseteq A_1$. Also, $x \in A_2$, $\exists \varepsilon_2 > 0$ s.t $B_{\varepsilon_2}(x) \subseteq A_2$.

Choose $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then $B_\varepsilon(x) \subseteq (A_1 \cap A_2)$.

(2) If $A_i \subseteq X$, $i \in I$ are open, then $\cup_{i \in I} A_i$ is open.

Let $x \in \bigcup_{i \in I} A_i$. $\exists i \in I$ s.t. $x \in A_i$. Since $A_i$ is open, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq A_i$. Since $A_i \subseteq \bigcup_{i \in I} A_i$, we are done

Using DeMorgan, we immediately have the following corollary:

## Corollary

*Let $(X, d)$ be a metric space.*

1. *Let $A_1, A_2 \subseteq X$. If $A_1$ and $A_2$ are closed, then $A_1 \cup A_2$ is closed.*
2. *If $A_i \subseteq X$, $i \in I$ are closed, then $\cap_{i \in I} A_i$ is closed.*

### Definition (Interior and closure)

Let $A \subseteq X$ where $(X, d)$ is a metric space.

- The *closure* of $A$ is $\overline{A} := \{x \in X : \forall \varepsilon > 0 \quad B_\varepsilon(x) \cap A \neq \emptyset\}$
- The *interior* of $A$ is $\mathring{A} := \{x \in A : \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq A\}$
- The *boundary* of $A$ is $\partial A := \{x \in X : \forall \varepsilon > 0,\ B_\varepsilon(x) \cap A \neq \emptyset \text{ and } B_\varepsilon(x) \cap A^c \neq \emptyset\}$

### Example

Let $X = (a, b] \subseteq \mathbb{R}$ with the ordinary (Euclidean) metric. Then

$$\overline{X} = [a, b], \quad \mathring{X} = (a, b), \quad \partial X = \{a, b\}$$

## Proposition

*Let $A \subseteq X$ where $(X, d)$ is a metric space. Then $\mathring{A} = A \setminus \partial A$.*

*Proof.* First, we show $\mathring{A} \subseteq A \setminus \partial A$.

Let $x \in \mathring{A}$. $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq A$. Clearly $x \in A$. Also $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \cap A^c = \emptyset$. Thus $x \notin \partial A$, so $x \in A \setminus \partial A$.

Next, $A \setminus \partial A \subseteq \mathring{A}$. Let $x \in A \setminus \partial A$. Then $x \in A$ and $x \notin \partial A$. $x \notin \partial A$ means $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \cap A = \emptyset$ or $B_\varepsilon(x) \cap A^c = \emptyset$. Since $x \in A$, the first is false, so $B_\varepsilon(x) \cap A^c = \emptyset \Rightarrow B_\varepsilon(x) \subseteq A$. Thus $x \in \mathring{A}$.

## Proposition

*Let $(X, d)$ be a metric space and $A \subseteq X$. $\overline{A}$ is closed and $\mathring{A}$ is open.*

*Proof.*

$\mathring{A}$ is open by definition.

We show $\overline{A}$ closed by showing $\overline{A}^c$ is open. Let $x \in \overline{A}^c$. Then $\exists \tilde{\varepsilon} > 0$ such that $B_{\tilde{\varepsilon}}(x) \cap A = \emptyset$.

Let $y \in B_{\tilde{\varepsilon}}(x)$. $y \in \overline{A}^c$ by definition. Therefore $B_{\tilde{\varepsilon}}(x) \subseteq \overline{A}^c$ so $\overline{A}^c$ is open.

## Remark

$\ast$ In fact, $\mathring{A} = \bigcup\{U : U \text{ is open and } U \subseteq A\}$ and $\overline{A} = \bigcap\{F : F \text{ is closed and } A \subseteq F\}$. $\ast$

# Sequences

**Definition (Sequence)**

Let $(X, d)$ be a metric space. A *sequence* is an ordered list of points $x_n$, $n \in \mathbb{N}$, in $X$, denoted $(x_n)_{n\in\mathbb{N}}$. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ *converges* to a point $x \in X$ if

$$\forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } d(x_n, x) < \varepsilon \quad \forall n \geq n_\varepsilon$$

**Recall:** $\overline{A} = \{x \in X : \forall \varepsilon > 0 \quad B_\varepsilon(x) \cap A \neq \emptyset\}$

## Proposition

Let $(X, d)$ be a metric space, and let $A \subseteq X$. Then $\overline{A}$ is equal to the set of points in $X$ which are limits of a sequence in $A$.

*Proof.*

$\Rightarrow$ Let $x \in \overline{A}$. Then by definition, $\forall \varepsilon > 0 \quad B_\varepsilon(x) \cap A \neq \emptyset$.
In particular, this is true for $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$.

For any $n \in \mathbb{N}$, we can choose $x_n \in A$ s.t. $x_n \in B_{1/n}(x)$.
Therefore $d(x, x_n) < \frac{1}{n}$. Since $\frac{1}{n} \downarrow 0$ monotonically,
$x_n \to x$.

$(\Leftarrow)$ Let $x \in X$ be the limit of a sequence $(x_n)_{n \in \mathbb{N}}$ in $A$. $\forall \varepsilon > 0$ $\exists n_\varepsilon \in \mathbb{N}$ s.t. $d(x_n, x) < \varepsilon$ $\forall n \geq n_\varepsilon$.

This means $\forall \varepsilon > 0$, $\exists x_n \in A$ s.t. $x_n \in B_\varepsilon(x)$.

$\therefore \forall \varepsilon > 0$, $A \cap B_\varepsilon(x) \neq \emptyset$. $\therefore x \in \overline{A}$.

## Corollary

A set $F \subseteq X$, where $(X, d)$ is a metric space, is closed if and only if every sequence in $F$ which converges in $X$ converges to a point in $F$.

**Remark:**

$F \subseteq X$ is closed $\iff F = \overline{F}$

# Cluster points of a set

**Definition**

Let $(X, d)$ be a metric space and $A \subseteq X$. A point $x \in X$ is a *cluster point* of $A$ (also called accumulation point) if for every $\epsilon > 0$, $B_\epsilon(x)$ contains infinitely many points in $A$.

## Proposition

$x \in X$ is a cluster point of $A \subseteq X$ where $(X, d)$ is a metric space if and only if there exists a sequence of points $x_n \in A$, $n \in \mathbb{N}$, such that $x_n \to x$.

*Proof.*

($\Leftarrow$) Suppose $\exists$ a sequence $(x_n)_{n\in\mathbb{N}}$ in $A$ s.t. $x_n \to x$.
Then $\forall \varepsilon > 0$, $B_\varepsilon(x)$ contains infinitely many elements of the sequence $(x_n)_{n\in\mathbb{N}}$. Since the $x_n \in A$, $x$ is a cluster point of $A$.

($\Rightarrow$) Suppose $x$ is a cluster point of $A$. Then $\forall \varepsilon > 0$, $\exists x_\varepsilon \in A$ s.t. $x_\varepsilon \in B_\varepsilon(x)$. Take $\varepsilon = \frac{1}{n}$.
$\exists x_n \in A$ s.t. $x_n \in B_{1/n}(x)$. By construction, $x_n \to x$.

Combining the previous result with the limit characterization of closure gives the following:

**Corollary**

For $A \subseteq X$, $(X, d)$ a metric space, we have

$$\overline{A} = A \cup \{x \in X : x \text{ is a cluster point of } A\}.$$

# Cauchy sequences

**Definition (Cauchy sequence)**

Let $(X, d)$ be a metric space. A sequence denoted $(x_n)_{n \in \mathbb{N}} \in X$ is called a *Cauchy sequence* if

$$\forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } d(x_n, x_m) < \varepsilon \ \forall n, m \geq n_\varepsilon$$

## Proposition

Let $(X, d)$ be a metric space, and let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in $X$. Then $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

*Proof.* Let $\varepsilon > 0$. Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in $X$ which converges to some $x \in X$. Then $\exists n_\varepsilon \in \mathbb{N}$ s.t. $d(x, x_n) < \varepsilon/2 \ \forall n \geq n_\varepsilon$.

Let $n, m \geq n_\varepsilon$. By the triangle inequality,

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore (x_n)_{n\in\mathbb{N}}$ is Cauchy

## Definition

A metric space where every Cauchy sequence converges (to a point in the space) is called *complete*.

**Example:** $\mathbb{R}$, $\mathbb{R}^n$ with usual metrics are complete

## Proposition

Let $(X, d)$ be a metric space, and let $Y \subseteq X$.

(i) If $X$ is complete and if $Y$ is closed in $X$, then $Y$ is complete.

(ii) If $Y$ is complete, then it is closed in $X$.

*Proof.* (i) Let $X$ be a complete metric space and $Y \subseteq X$ be closed. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $Y$. Since $Y \subseteq X$, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $X$.
$\therefore (x_n)_{n\in\mathbb{N}}$ has to converge $x \in X$ since $X$ is complete.
Since $Y$ is closed, we must have $x \in Y$.
$\therefore Y$ is complete.

(ii) Let $Y \subseteq X$ be complete. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $Y$ that converges to some $y \in X$. $(y_n)_{n\in\mathbb{N}}$ is Cauchy in $X$ (and in $Y$). Since $Y$ is complete, $(y_n)_{n\in\mathbb{N}}$ converges to $y' \in Y$. Since sequences converge to a unique point, $y = y' \in Y$, so $Y$ is closed.

# Subsequences

## Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space $(X, d)$. Let $(n_k)_{k\in\mathbb{N}}$ be a sequence of natural numbers with $n_1 < n_2 < \cdots$. The sequence $(x_{n_k})_{k\in\mathbb{N}}$ is called a *subsequence* of $(x_n)_{n\in\mathbb{N}}$. If $(x_{n_k})_{k\in\mathbb{N}}$ converges to $x \in X$, we call $x$ a *subsequential limit*.

## Example

$((-1)^n)_{n\in\mathbb{N}} = \{-1, 1, -1, 1, -1, 1, -1, \cdots\}$

This sequence <u>diverges</u>. The subsequences $((-1)^{2n})_{n\in\mathbb{N}}$ and $((-1)^{2n-1})_{n\in\mathbb{N}}$ converge to $1$ and $-1$, respectively.

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## Proposition

A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space $(X, d)$ converges to $x \in X$ if and only if every subsequence of $(x_n)_{n\in\mathbb{N}}$ also converges to $x$.

*Proof.*

($\Leftarrow$) Suppose that every subsequence of $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$. Since $(x_n)_{n\in\mathbb{N}}$ is a subsequence of itself, $(x_n)_{n\in\mathbb{N}}$ converges to $x$.

($\Rightarrow$) Suppose $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$. Let $(x_{n_k})_{k\in\mathbb{N}}$ be an arbitrary subsequence of $(x_n)_{n\in\mathbb{N}}$.

*Proof continued* Let $\varepsilon > 0$ be arbitrary. $\exists n_\varepsilon \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ $\forall n \geq n_\varepsilon$. Choose $k_\varepsilon$ such that $n_{k_\varepsilon} \geq n_\varepsilon$. This must exist since $(n_k)_{k \in \mathbb{N}}$ is strictly increasing.

Then $\forall k \geq k_\varepsilon$, $d(x_{n_k}, x) < \varepsilon$.

$$\therefore x_{n_k} \to x$$

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