Module 5: Metric spaces III

Operational math bootcamp

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Last time

Looked at open and closed sets.

Discussed sequences, which includes Cauchy sequences and subsequences:

- Convergent sequence: $x_n \to x \Leftrightarrow \forall \epsilon > 0 \, \exists \, n_\epsilon \in \mathbb{N}$ s.t. $d(x_n, x) < \epsilon$ for all $n \geq n_\epsilon$
- Cauchy sequence: $\forall \epsilon > 0 \, \exists \, n_\epsilon \in \mathbb{N}$ s.t. $d(x_n, x_m) < \epsilon$ for all $n, m \geq n_\epsilon$
- Proved that a convergent sequence is Cauchy
- Discussed complete metric spaces, where all Cauchy sequences converge (like $\mathbb{R}$ with the usual metric, absolute value)
- Proved that a sequence converges to x if and only if all subsequences converge to x

Outline for today

- Continuity
- Equivalent metrics
- Density and separability

Continuity

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$, and let $f : X \to Y$. f is *continuous* at x_0 if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X that converges to x_0, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

We say that f is continuous if it is continuous at every point in X.

Theorem

Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$, and let $f : X \to Y$. The following are equivalent:

(i) *f is continuous at x_0* (previous definition)

(ii) *for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0))) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$*

(iii) *for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$*

(i) f is continuous at x_0 : every sequence (x_n) in X with $x_n \to x_0$, $\lim_{n\to\infty} f(x_n) = f(x_0)$
(ii) for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0))) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$
(iii) for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$

Proof. (i) $\Rightarrow$ (ii) We prove the contrapositive. $\neg(ii) \Rightarrow \neg(i)$

$\exists \varepsilon_0 > 0$ s.t $\forall \delta > 0$ $\exists x_\delta \in X$ with $d_X(x_0, x_\delta) < \delta$ but (*)

(**) $d_Y(f(x_0), f(x_\delta)) \geq \varepsilon_0$. We need to find a sequence in X that converges to x_0 but $f(x_n) \not\to f(x_0)$.

Let $\delta = \frac{1}{n}$, $n \in \mathbb{N}$. We pick our sequence using (*), which converges to x_0. Then $d_Y(f(x_0), f(x_n)) \geq \varepsilon_0$ by (**) $\forall n \in \mathbb{N}$

$\therefore \lim_{n\to\infty} f(x_n) \neq f(x_0)$.

(ii) $\Rightarrow$ (iii) Follows from the definition of open ball and pre-image (exercise)

(iii) $\Rightarrow$ (i) (iii) $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0)))$

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X that converges to x_0. Let $\varepsilon > 0$ be arbitrary. By (iii), $\exists \delta > 0$ s.t. $B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0)))$. If $x \in B_\delta(x_0)$, then $d_Y(f(x_0), f(x)) < \varepsilon$. Since $(x_n)_{n\in\mathbb{N}}$ converges to x_0, $\exists n_0 \in \mathbb{N}$ s.t. $d(x_n, x) < \delta \ \forall n \geq n_0$.

Therefore, $d_Y(f(x_0), f(x)) < \varepsilon \ \forall n \geq n_0$ by (iii).

$\therefore f(x_n) \to f(x_0)$.

Corollary

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. The following are equivalent:

(i) f is continuous

(ii) if $U \subseteq Y$ is open, then $f^{-1}(U)$ is open

(iii) if $F \subseteq Y$ is closed, then $f^{-1}(F)$ is closed

We need the following results about sets and functions:
Let X and Y be sets and $f : X \to Y$. Let $A, B \subseteq Y$. Then

(*) 1. $A \subseteq B \implies f^{-1}(A) \subseteq f^{-1}(B)$
2. $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$

(exercise)

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.

(i) $\Rightarrow$ (ii): Suppose f is continuous (at every point $x \in X$).
Let $U \subseteq Y$ be open. Let $x \in f^{-1}(U)$. Then $f(x) \in U$.
Since U is open, $\exists \varepsilon_0 > 0$ s.t. $B_{\varepsilon_0}(f(x)) \subseteq U$.
By (iii) from the previous theorem, $\exists \delta_0 > 0$ s.t.
$B_{\delta_0}(x) \subseteq f^{-1}(B_{\varepsilon_0}(f(x)))$. Since $B_{\varepsilon_0}(f(x)) \subseteq U$,
$\Rightarrow f^{-1}(B_{\varepsilon_0}(f(x))) \subseteq f^{-1}(U)$ by (*).

Thus for each $x \in f^{-1}(U)$, $\exists \delta_0 > 0$ s.t.

$$B_{\delta_0}(x) \subseteq f^{-1}(B_{\varepsilon_0}(f(x))) \subseteq f^{-1}(U).$$

$\therefore f^{-1}(U)$ is open.

(ii) $\Rightarrow$ (i)

Let's use the definition of continuity (iii):

for $x \in X$, $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$

Let $x \in X$ and $\varepsilon > 0$. Since $B_\varepsilon(f(x))$ is open, so by (ii), $f^{-1}(B_\varepsilon(f_x))$ is also open. Since $x \in f^{-1}(B_\varepsilon(f(x)))$, by the definition of open set, $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$.

We're done.

(ii) $\Rightarrow$ (iii) Let $F \subseteq Y$ be closed.

$\Rightarrow Y \setminus F$ is open

$\Rightarrow$ by (ii), $f^{-1}(Y \setminus F)$ is open

Since $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$,

$f^{-1}(F)$ is closed

(iii) $\Rightarrow$ (ii)

exercise

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.

- f is *uniformly continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that for every $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$, we have $d_Y(f(x_1), f(x_2))) < \epsilon$
- f is *Lipschitz continuous* if there exists a $K > 0$ such that for every $x_1, x_2 \in X$ we have $d_Y(f(x_1), f(x_2))) \leq K d_X(x_1, x_2)$

Proposition

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.

$$f \text{ is Lipschitz continuous} \;\Rightarrow\; \text{f is uniformly continuous} \;\Rightarrow\; \text{f is continuous}$$

Proof is one of your exercises.

Contraction Mapping Theorem

Definition

Let (X, d) be a metric space and let $f : X \to X$. We say that $x^* \in X$ is a *fixed point* of f if $f(x^*) = x^*$.

Definition

Let (X, d) be a metric space and let $f : X \to X$. f is a *contraction* if there exists a constant $k \in [0, 1)$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq kd(x, y)$.

Observe that a function is a contraction if and only if it is Lipschitz continuous with constant $K < 1$.

Theorem (Contraction Mapping Theorem)

Suppose that $f : X \to X$ is a contraction and the metric space X is complete. Then f has a unique fixed point x^*.

Example

Let $f : \left[-\frac{1}{3}, \frac{1}{3}\right] \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ be defined by the mapping $x \mapsto x^2$. Assume we use the standard Euclidean metric, $d(x, y) = |x - y|$. f has a unique fixed point because

(1) $\left[-\frac{1}{3}, \frac{1}{3}\right]$ is a complete metric space with $|\cdot|$

(2) Let $x, y \in \left[-\frac{1}{3}, \frac{1}{3}\right]$.

Then $|x^2 - y^2| = |x+y||x-y| \leq \frac{2}{3}|x-y|$

$\therefore$ f is a contraction with $K = 2/3$

Equivalent metrics

Definition (Equivalent metrics)

Two metrics d_1 and d_2 on a set X are *equivalent* if the identity maps from (X, d_1) to (X, d_2) and from (X, d_2) to (X, d_1) are continuous.

$1_X : X \mapsto X$

Proposition

Two metrics d_1, d_2 on a set X are equivalent if and only if they have the same open sets or the same closed sets.

Definition

Two metrics d_1 and d_2 on a set X are *strongly equivalent* if for every $x, y \in X$, there exists constants $\alpha > 0$ and $\beta > 0$ such

$$\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y).$$

If two metrics are strongly equivalent then they are equivalent. The proof of this is one of the exercises.

Example

We show that the Euclidean distance (induced by 2-norm) and the metric induced by the ∞-norm are equivalent on $\mathbb{R}^n$.

$$\|x-y\|_2 = \sqrt{\sum_{j=1}^{n}(x_j-y_j)^2}, \quad \|x-y\|_\infty = \max_{j=1,\dots,n}|x_j-y_j|$$

$$\|x-y\|_2 = \sqrt{\sum_{j=1}^{n}(x_j-y_j)^2} \le \sqrt{n\max_{j=1,\dots,n}(x_j-y_j)^2} = \sqrt{n}\max_{j=1,\dots,n}|x_j-y_j| = \sqrt{n}\,\|x-y\|_\infty$$

$$\|x-y\|_\infty = \max_{j=1,\dots,n}|x_j-y_j| = \sqrt{\max_{j=1,\dots,n}(x_j-y_j)^2} \le \sqrt{\sum_{j=1}^{n}(x_j-y_j)^2} = \|x-y\|_2$$

Can you think of an example that we've seen of a metric that isn't equivalent to the Euclidean metric? exercise

Density

Definition

Let (X, d) be a metric space. A subset $A \subseteq X$ is called *dense* if $\overline{A} = X$.

Using the definition of closure, $A \subseteq X$ is dense if and only if $\forall x \in X, \forall \varepsilon > 0, \; B_\varepsilon(x) \cap A \neq \emptyset$.

Why are dense sets important?

We can look at the dense subset instead of the whole space (use it for approximations)

Examples

1. $(\mathbb{R}, |\cdot|)$

$\mathbb{Q}$ is dense in $\mathbb{R}$ with $|\cdot|$

Archimedean prop: $\forall x, y \in \mathbb{R}$, $x > 0$, $\exists n \in \mathbb{N}$ s.t. $nx > y$

2. Let X be a set and define $d\colon X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

The only dense set in X is X itself.

(exercise)

Definition

A metric space (X, d) is *separable* if it contains a countable dense subset.

Example:

$\mathbb{R}$ is separable since $\mathbb{Q}$ is dense in $\mathbb{R}$

Example

Define $\ell_\infty = \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{R}, \sup_{n\in\mathbb{N}} |x_n| < \infty\}$, the space of bounded real valued sequences. Endow ℓ_∞ with a metric induced by the supremum norm, namely $d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sup_{n\in\mathbb{N}} |x_n - y_n|$. Then ℓ_∞ is **not separable** with respect to the topology induced by this metric.

Note: this proof relies on content from the cardinality section. Specifically, Cantor's theorem gives us that $|A| < |\mathcal{P}(A)|$ for any set A.

Proof.

References

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Runde, Volker (2005). *A Taste of Topology*. Universitext. url: https://link.springer.com/book/10.1007/0-387-28387-0