Exercises for Module 6: Metric Spaces IV

1. Let X be a set and define $d\colon X \times X \to [0, \infty)$ by

$$d(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

Show that $S \subseteq X$ is compact if and only if S is finite.

We showed in class that S finite $\Rightarrow$ S is compact for any metric space (X,d).

We show S compact $\Rightarrow$ S finite with the discrete metric.

Assume S is compact. Then any open cover for S has a finite subcover. Let $\varepsilon \in (0,1)$. Then $\{B_\varepsilon(x)\}_{x\in S}$ is an open cover for S and $\exists n \in \mathbb{N}$ s.t $S \subseteq \bigcup_{i=1}^{n} B_\varepsilon(x_i)$. Recall that $B_\varepsilon(x) = \{x\}$ for $0<\varepsilon<1$. Thus $S \subseteq \bigcup_{i=1}^{n} \{x_i\} \Rightarrow S \subseteq \{x_1, \dots, x_n\}$

$\therefore$ S is finite

2. Let (X,d) be a metric space and $K \subset X$ compact. Show that for all $\epsilon > 0$ there exists $\{x_1, x_2, \dots, x_n\} \subseteq K$ such that for all $y \in K$ we have $d(y, x_i) < \epsilon$ for some $i = 1, \dots, n$.

Let $\varepsilon > 0$ and $y \in K$, $K \subset X$ compact. Since K is compact and $\{B_\varepsilon(x)\}_{x \in K}$ is an open cover for K, $\exists n \in \mathbb{N}$ s.t. $K \subseteq \bigcup_{i=1}^{n} B_\varepsilon(x_i) \Rightarrow y \in \bigcup_{i=1}^{n} B_\varepsilon(x_i)$

$\Rightarrow \exists i \in \{1, \dots, n\}$ s.t. $y \in B_\varepsilon(x_i)$

$\Rightarrow \exists i \in \{1, \dots, n\}$ s.t. $d(x_i, y) < \varepsilon$

3. Define the sequence $(a_n)_{n\in\mathbb{N}}$ by $a_1 = 2$ and

$$a_{k+1} = \frac{a_k + 5}{3}, \qquad k \geq 1.$$

Determine if the limit $\lim_{n\to\infty} a_n$ exists and, if so, calculate it.

Claim: $a_k \leq a_{k+1}$ $\forall k \geq 1$

Proof by induction. Base case: $a_1 = 2$, $a_2 = \frac{7}{3} < 2 = a_1$

Suppose the claim holds for some $n \geq 1$. Then

$$a_{n+1} \geq a_n \Rightarrow a_{n+1} + 5 \geq a_n + 5$$
$$\Rightarrow \frac{a_{n+1}+5}{3} \geq \frac{a_n+5}{3}$$
$$\Rightarrow a_{n+2} \geq a_{n+1}$$

$\therefore$ The claim holds by induction.

Claim: $a_k < \frac{5}{2}$ $\forall k \geq 1$

By induction. Base case: $a_1 = 2 < 5/2$.

IH: Assume $a_n < \frac{5}{2}$ for some $n \geq 1$.

Then $a_{n+1} = \frac{a_n+5}{3} < \frac{\frac{5}{2}+5}{3} = \frac{15}{6} = \frac{5}{2}$

$\therefore$ The claim holds by induction.

Therefore $(a_k)_{k\in\mathbb{N}}$ is a bounded, monotone sequence. By the monotone convergence theorem, $\lim_{k\to\infty} a_k$ must exist.

We calculate: $\lim_{k\to\infty} a_{k+1} = \lim_{k\to\infty} \frac{a_k+5}{3}$

$$\Rightarrow \lim_{k\to\infty} a_{k+1} = \frac{1}{3}\lim_{k\to\infty} a_k + \frac{5}{3}$$

$$\Rightarrow \lim_{k\to\infty} a_{k+1} = \frac{5}{2}.$$

4. Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ be bounded sequences in $\mathbb{R}$. Show that

$$\limsup_{n\to\infty}(x_n+y_n) \le \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$

and give an example where the inequality is strict.

<u>Claim:</u> $\sup\limits_{k\ge n}(x_k+y_k) \le \sup\limits_{k\ge n} x_k + \sup\limits_{k\ge n} y_k$

Let $\overline{x}_n = \sup\limits_{k\ge n} x_k$, $\overline{y}_n = \sup\limits_{k\ge n} y_k$. Then $\overline{x}_n \ge x_k \ \forall k\ge n$ and $\overline{y}_n \ge y_k \ \forall k \ge n$. Therefore $\overline{x}_n + \overline{y}_n \ge x_k + y_k \ \forall k\ge n$

$\Rightarrow \overline{x}_k + \overline{y}_k$ is an upper bound for $\{x_k+y_k : k\ge n\}$

$\Rightarrow \sup\limits_{k\ge n}(x_k+y_k) \le \overline{x}_n + \overline{y}_n = \sup\limits_{k\ge n} x_k + \sup\limits_{k\ge n} y_k$

since the supremum is the least upper bound

Then $\limsup\limits_{n\to\infty}(x_n+y_n) = \lim\limits_{n\to\infty}\sup\limits_{k\ge n}(x_k+y_k) \le \lim\limits_{n\to\infty}\sup\limits_{k\ge n} x_k + \sup\limits_{k\ge n} y_k$

$= \limsup\limits_{n\to\infty} x_n + \limsup\limits_{n\to\infty} y_n$

<u>Example</u> where $<$ is strict: $x_n = (-1)^n$, $y_n = (-1)^{n+1}$, $n\in\mathbb{N}$

$\limsup\limits_{n\to\infty}(x_n+y_n) = 0 < 2 = \limsup\limits_{n\to\infty} x_k + \limsup\limits_{n\to\infty} y_k$

5. Let $(x_n)_{n\in N}$ be a sequence in $\mathbb{R}$. Show that $\lim_{n\to\infty} x_n = 0$ if and only if $\limsup_{n\to\infty}|x_n| = 0$.

($\Rightarrow$) Suppose $\lim\limits_{n\to\infty} x_n = 0$. Then by the theorem in lecture, $\limsup\limits_{n\to\infty} x_n = 0$. This implies $\limsup\limits_{n\to\infty}|x_n| = 0$.

($\Leftarrow$) Suppose $\limsup\limits_{n\to\infty}|x_n| = 0$.

Since $\limsup\limits_{n\to\infty}|x_n| \ge \liminf\limits_{n\to\infty}|x_n|$ and $|x_n|\ge 0 \ \forall n \Rightarrow \liminf\limits_{n\to\infty}|x_n| \ge 0$, we have

$0 = \limsup\limits_{n\to\infty}|x_n| \ge \liminf\limits_{n\to\infty}|x_n| \ge 0$

$\Rightarrow \limsup\limits_{n\to\infty}|x_n| = 0 = \liminf\limits_{n\to\infty}|x_n|$

$\Rightarrow \lim\limits_{n\to\infty}|x_n| = 0$ by theorem from class

$\Rightarrow \lim\limits_{n\to\infty} x_n = 0$

<u>Lemma</u> $x_n \to 0 \iff |x_n| \to 0$

($\Rightarrow$) Suppose $x_n\to 0$. Then $\forall \varepsilon>0 \ \exists n_\varepsilon \in \mathbb{N}$ s.t. $|x_n - 0| < \varepsilon \ \forall n \ge n_\varepsilon$, i.e. $|x_n| < \varepsilon \ \forall n\ge n_\varepsilon$.
Let $\varepsilon > 0$ arbitrary. Choose $n = n_\varepsilon$, then $\forall n \ge n_\varepsilon$, $||x_n| - 0| = |x_n| < \varepsilon$ as required.

($\Leftarrow$) Similar.