Exercises for Module 6: Metric Spaces IV

1. Let X be a set and define $d\colon X \times X \to [0, \infty)$ by

$$d(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

Show that $S \subseteq X$ is compact if and only if S is finite.

2. Let (X, d) be a metric space and $K \subset X$ compact. Show that for all $\epsilon > 0$ there exists $\{x_1, x_2, \ldots, x_n\} \subseteq K$ such that for all $y \in K$ we have $d(y, x_i) < \epsilon$ for some $i = 1, \ldots, n$.

3. Define the sequence $(a_n)_{n\in\mathbb{N}}$ by $a_1 = 2$ and

$$a_{k+1} = \frac{a_k + 5}{3}, \qquad k \geq 1.$$

Determine if the limit $\lim_{n\to\infty} a_n$ exists and, if so, calculate it.

4. Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ be bounded sequences in $\mathbb{R}$. Show that

$$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$

and give an example where the inequality is strict.

5. Let $(x_n)_{n\in N}$ be a sequence in $\mathbb{R}$. Show that $\lim_{n\to\infty} x_n = 0$ if and only if $\limsup_{n\to\infty} |x_n| = 0$.