Module 6: Metric Spaces IV

Operational math bootcamp

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Outline

- Compactness
- Extra properties of $\mathbb{R}$
  - Right- and left-continuity
  - Lim sup and lim inf

Last time

Definition

Let (X, d) be a metric space. A subset $A \subseteq X$ is called *dense* if $\overline{A} = X$.

Definition

A metric space (X, d) is *separable* if it contains a countable dense subset.

Example

$\mathbb{R}$ is separable because $\mathbb{Q}$ is dense in $\mathbb{R}$

Example

Define $\ell_\infty = \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{R}, \sup_{n\in\mathbb{N}} |x_n| < \infty\}$, the space of bounded real valued sequences. Endow ℓ_∞ with a metric induced by the supremum norm, namely $d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sup_{n\in\mathbb{N}} |x_n - y_n|$. Then ℓ_∞ is **not separable** with respect to the topology induced by this metric.

Note: this proof relies on content from the cardinality section. Specifically, Cantor's theorem gives us that $|A| < |\mathcal{P}(A)|$ for any set A.

Proof. For $M \subseteq \mathbb{N}$, define $e_n^M = \begin{cases} 1 & \text{if } n \in M \\ 0 & \text{otherwise} \end{cases}$. (sequence)

If $M_1, M_2 \subseteq \mathbb{N}$ are non-empty and $M_1 \neq M_2$, then $d\left((e_n^{M_1})_{n\in\mathbb{N}}, (e_n^{M_2})_{n\in\mathbb{N}}\right) = 1$. The open balls $B_{1/3}\left((e_n^{M_1})_{n\in\mathbb{N}}\right)$, $B_{1/3}\left((e_n^{M_2})_{n\in\mathbb{N}}\right)$ are disjoint.

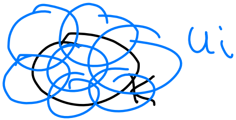
Proof continued. Suppose in order to derive a contradiction that $\exists A \subseteq \ell^\infty$ that is dense & countable.

A is dense $\Rightarrow$ $\exists (x_n)_{n\in\mathbb{N}} \in \ell^\infty$ and $\forall \varepsilon > 0$, $B_\varepsilon((x_n)_{n\in\mathbb{N}}) \cap A \neq \emptyset$.

In particular, for all $M \subseteq \mathbb{N}$, $B_{1/3}((e_n^M)_{n\in\mathbb{N}}) \cap A \neq \emptyset$.

There are uncountably many such M ($\mathcal{P}(\mathbb{N})$ is uncountable), but we assumed A has countably many elements. Since the balls are disjoint, this is a contradiction.

Compactness

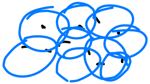


Definition

Let (X, d) be a metric space and $K \subseteq X$.

A collection $\{U_i\}_{i\in I}$ of open sets is called *open cover* of K if $K \subseteq \cup_{i\in I} U_i$.

The set K is called *compact* if for all open covers $\{U_i\}_{i\in I}$ there exists a finite subcover, meaning there exists an $n \in \mathbb{N}$ and $\{U_1, \ldots, U_n\} \subseteq \{U_i\}_{i\in I}$ such that $K \subseteq \cup_{i=1}^{n} U_i$.



Example

Let $S \subseteq X$ where (X, d) is a metric space. If S is finite, then it is compact.

Proof.

Since S is finite, we can write $S = \{x_1, x_2, \dots, x_n\}$. Let $\{U_i\}_{i \in I}$ be an open cover for S. Then $\exists j = 1, \dots, n$ such that $x_i \in U_j$. Then $S \subseteq \bigcup_{j=1}^{n} U_j$.

S is compact

Example

$(0, 1)$ is not compact.

Proof.

The set $\{U_n\}_{n\in\mathbb{N}}$ defined by $U_n = (\frac{1}{n}, 1)$ is an open for $(0,1)$ because $(0,1) \subseteq \bigcup_{n=1}^{\infty} U_n$.

Suppose in order to derive a contradiction that there exists a finite subcover, meaning $\exists n^* \in \mathbb{N}$ s.t. $(0,1) \subseteq \bigcup_{j=1}^{j^*} (\frac{1}{n_j}, 1)$. This means $(0,1) \subseteq (\frac{1}{n_{j^*}}, 1)$ since the sets are nested. But $\exists x \in (0,1)$ s.t. $0 < x < \frac{1}{n_{j^*}}$ for any n_{j^*}. Contradiction.

$\forall x, y > 0$, $\exists n \in \mathbb{N}$ s.t $nx > y$ (Archimedean property)

Proposition

Let (X, d) be a metric space and take a non-empty subset $K \subseteq X$. The following holds:

1. If X is compact and K is closed, then K is compact (i.e. closed subsets of compact sets are compact).
2. If K is compact, then K is closed.

Proof. (1) If X is compact and $K \subseteq X$ is closed, then K is compact

We want to show that any open cover of K has a finite subcover. Let $\{U_i\}_{i \in I}$ be an open cover for K. Since K^c is open, $\{U_i\}_{i \in I} \cup K^c$ is an open cover for X. Since X is compact, there exist a finite subcover of the form $\bigcup_{j=1}^{n} U_j$ or $\left(\bigcup_{j=1}^{n} U_j\right) \cup K^c$. Either way, $\bigcup_{j=1}^{n} U_j$ is a finite subcover for K. $\therefore K$ is compact.

(2) $K \subseteq X$ compact $\Rightarrow K$ is closed.

We show K^c is open. Let $x \in K^c$. By properties of a metric, $\forall y \in K$, $d(x,y) > 0$. Let $\tilde{\varepsilon} = d(x,y)$.

Then $B_{\frac{\tilde{\varepsilon}}{2}}(x)$ and $B_{\frac{\tilde{\varepsilon}}{2}}(y)$ are disjoint.

Since K is compact and $\{B_{\frac{\tilde{\varepsilon}}{2}}(y)\}_{y \in K}$ is an open cover, there exist $i=1,\dots,n$ such that $K \subseteq \bigcup_{i=1}^{n} B_{\frac{\tilde{\varepsilon}}{2}}(y_i)$

Define $U_x = \bigcap_{i=1}^{n} B_{\frac{d(x,y_i)}{2}}(x)$. U_x contains x, is open, is disjoint from the subcover for K.

$\Rightarrow U_x \subseteq K^c \Rightarrow K^c$ is open $\Rightarrow K$ is closed

Arbitrary compact metric spaces have some nice properties in general as the next proposition shows.

Proposition

A compact metric space (X, d) is complete and separable.

Also, just as we had a sequential characterization of the closure of a set in metric spaces, we similarly have a sequential characterization of compactness.

Theorem

Let (X, d) be a metric space. Then $K \subseteq X$ is compact with respect to the metric induced by d if and only if every sequence in K admits a subsequence converging to some point in K.

Compactness on $\mathbb{R}^n$

Theorem (Heine-Borel Theorem)

Let $K \subseteq \mathbb{R}^n$. Then K is compact with respect to the topology induced by the Euclidean distance if and only if it is closed and bounded.

A corollary of the last two theorems is the Bolzano-Weierstrass theorem.

Corollary (Bolzano-Weierstrass)

Any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Proposition

Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $K \subseteq X$ is compact and let $f : K \to Y$ be continuous. Then $f(K)$ is compact.

Note: this is a generalization of the Extreme Value Theorem to metric spaces.

$[a, b]$

Recall from the set theory section:
If $f : X \to Y$:

1. $A \subseteq B \subseteq Y \Rightarrow f^{-1}(A) \subseteq f^{-1}(B)$ and $A \subseteq B \subseteq X \Rightarrow f(A) \subseteq f(B)$
2. $f^{-1}(\cup_{i \in I} A_i) = \cup_{i \in I} f^{-1}(A_i)$, where $A_i \subseteq Y \; \forall i \in I$
3. $f(\cup_{i \in I} A_i) = \cup_{i \in I} f(A_i)$, where $A_i \subseteq X \; \forall i \in I$
4. $A \subseteq X \Rightarrow A \subseteq f^{-1}(f(A))$
5. $B \subseteq Y \Rightarrow f(f^{-1}(B)) \subseteq B$

Proof. Let $\{U_i\}_{i\in I}$ be an open cover for $f(K)$.

$\Rightarrow f(K) \subseteq \bigcup_{i\in I} U_i$ by definition

$\Rightarrow f^{-1}(f(K)) \subseteq f^{-1}\left(\bigcup_{i\in I} U_i\right)$ by ①

$= \bigcup_{i\in I} f^{-1}(U_i)$ by ②

By ④ $K \subseteq f^{-1}(f(K)) \Rightarrow K \subseteq \overbrace{\underbrace{\bigcup_{i\in I} f^{-1}(U_i)}_{\text{open by def of continuous func}}}^{\text{open cover}}$

$\Rightarrow \exists n \in \mathbb{N}$ s.t. $K \subseteq \bigcup_{i=1}^{n} f^{-1}(U_i)$

since K is compact

Then $f(K) \subseteq f\left(\bigcup_{i=1}^{n} f^{-1}(U_i)\right)$ by 5

$= \bigcup_{i=1}^{n} f(f^{-1}(U_i))$ by 3

$\subseteq \bigcup_{i=1}^{n} U_i$ by 5

$\therefore f(K)$ is compact

Extra properties of $\mathbb{R}$

Right and left continuous

Recall: $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|x_0 - y| < \delta$ implies $|f(x_0) - f(y)| < \epsilon$.

$\Leftrightarrow \quad x_0 - \delta < x < x_0 + \delta$

Definition

Let $f : \mathbb{R} \to \mathbb{R}$.

- f is *left continuous* at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$, such $|f(x_0) - f(x)| < \epsilon$ whenever $x_0 - \delta < x < x_0$.
- f is *right continuous* at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$, such $|f(x_0) - f(x)| < \epsilon$ whenever $x_0 < x < x_0 + \delta$.

We say that f is left continuous if it is left continuous at all points in the domain, and similar for right continuous.

Proposition

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if it is left and right continuous.

Proof.

exercise

Bounded sequences and monotone convergence

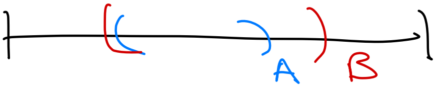
Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. We call $(x_n)_{n\in\mathbb{N}}$ *bounded* if there exists an $M > 0$ such that $|x_n| < M$ for all $n \in \mathbb{N}$.

Theorem (Monotone convergence theorem)

(i) *Suppose* $(x_n)_{n\in\mathbb{N}}$ *is an increasing sequence, i.e.* $x_n \leq x_{n+1}$ *for all* $n \in \mathbb{N}$, *and that it is bounded (above). Then the sequence converges. Furthermore,* $\lim_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} x_n$, *where* $\sup_{n\in\mathbb{N}} x_n := \sup\{x_n : \ n \in \mathbb{N}\}$.

(ii) *Suppose* $(x_n)_{n\in\mathbb{N}}$ *is a decreasing sequence, i.e.* $x_n \geq x_{n+1}$ *for all* $n \in \mathbb{N}$, *which is bounded (below). Then the sequence converges and* $\lim_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} x_n := \inf\{x_n : \ n \in \mathbb{N}\}$.



Convention: $\sup A = \infty$ if $A \subseteq \mathbb{R}$ is not bounded above and $\inf A = -\infty$ if A is not bounded below.

Lemma

If $A \subseteq B \subseteq \mathbb{R}$ *is non-empty, then* $\inf A \leq \sup A$, $\sup A \leq \sup B$, *and* $\inf A \geq \inf B$.

The proof of this follows from the definition of greatest lower and least upper bound.

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. We define the *limit superior* of $(x_n)_{n\in\mathbb{N}}$ as

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \sup_{k\geq n} x_k.$$

Similarly we define the *limit inferior* of $(x_n)_{n\in\mathbb{N}}$ as

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} \inf_{k\geq n} x_k.$$

If the sequence $(x_n)_{n\in\mathbb{N}}$ is not bounded above, then $\limsup_{n\to\infty} x_n = \infty$. Similarly, if the sequence $(x_n)_{n\in\mathbb{N}}$ is not bounded below, then $\liminf_{n\to\infty} x_n = -\infty$.

Proposition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$.

- The sequence of suprema, $s_n = \sup_{k\geq n} x_k$, is decreasing and the sequence of infima, $i_n = \inf_{k\geq n} x_k$, is increasing.
- The limit superior and the limit inferior of a bounded sequence always exist and are finite.

Proof. (1) is consequence of the lemma

(2) follows from Monotone Convergence Theorem

Theorem

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then the sequence converges to $x \in \mathbb{R}$ if and only if $\limsup_{n\to\infty} x_n = x = \liminf_{n\to\infty} x_n$.

Proof in notes.

We can extend this easily to a sequence of functions $f_n \colon X \to \mathbb{R}$ as follows:

Define $f = \limsup_{n\to\infty} f_n$ to be the function defined pointwise by $f(x) = \limsup_{n\to\infty}(f_n(x))$ and similar for the limit inferior.

There also exists a set theoretic version in terms of unions and intersections which you will encounter in probability.

References

Runde, Volker (2005). *A Taste of Topology*. Universitext. url: https://link.springer.com/book/10.1007/0-387-28387-0