

Module 8: Linear Algebra II

Operational math bootcamp



Statistical Sciences
UNIVERSITY OF TORONTO

Emma Kroell

University of Toronto

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Outline

Last time:

- Vector spaces and subspaces
- Linear independence and bases
- Linear maps, null space, range

Today:

- Inverses of linear maps
- Matrices as linear maps
- Determinants
- Inner product spaces

Recall

Theorem

Suppose $\mathbf{u}_1, \dots, \mathbf{u}_n$ is a basis for U and $\mathbf{v}_1, \dots, \mathbf{v}_n$ is a basis for V . Then there exists a unique linear map $T : U \rightarrow V$ such that $T\mathbf{u}_j = \mathbf{v}_j$ for $j = 1, \dots, n$.

Definition (Injective and surjective)

Let $T : U \rightarrow V$. T is **injective** if $T\mathbf{u} = T\mathbf{v}$ implies $\mathbf{u} = \mathbf{v}$, if and only if $\text{null } T = \{\mathbf{0}\}$.
 T is **surjective** if $\forall \mathbf{v} \in V, \exists \mathbf{u} \in U$ such that $\mathbf{v} = T\mathbf{u}$, i.e. if $\text{range } T = V$.

Theorem (Rank Nullity Theorem)

Let $T : U \rightarrow V$ be a linear transformation, where U and V are finite-dimensional vector spaces. Then

$$\text{rank } T + \text{nullity } T = \dim U.$$

$$\dim(\text{range } T) + \dim(\text{null } T) = \dim U$$

$$S : U \rightarrow V, T : V \rightarrow W$$

Definition (Product of linear maps)

Let $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$. We define the **product** $ST \in \mathcal{L}(U, W)$ for $\mathbf{u} \in U$ as $ST(\mathbf{u}) = S(T(\mathbf{u}))$.

Definition

A linear map $T : U \rightarrow V$ is **invertible** if there exists a linear map $S : V \rightarrow U$ such that ST is the identity map on U and TS is the identity map on V . Such a map S is called the *inverse* of T .

If T is invertible, we denote the inverse by T^{-1} . This is justified by the fact that the inverse is unique, as the next proposition shows.

$$TT^{-1} = I = T^{-1}T$$

Proposition

Any invertible linear map has a unique inverse.

Proof. Let $T: U \rightarrow V$. Suppose $S_1, S_2: V \rightarrow U$ are both inverses for T .

$$S_1 = S_1 I = S_1 T S_2 = I S_2 = S_2$$

S₂ is inverse

$\therefore S_1 = S_2$, i.e. inverse is unique

Theorem

A linear map is *invertible* if and only if it is *injective* and *surjective*.

See proof in the book.

Definition

An invertible linear map is called an *isomorphism*. If there exists an isomorphism from one vector space to another, we say that *the vector spaces are isomorphic*.

Theorem

Two finite-dimensional vector spaces over \mathbb{F} are isomorphic if and only if they have the same dimension.

Proof. (\Rightarrow) Suppose \exists linear map $T: U \rightarrow V$, U, V are vector spaces (finite-dim) which is invertible.

By the previous Theorem, T is invertible $\Rightarrow T$ is injective and surjective. $\Rightarrow \text{null } T = \{\vec{0}\}$, $\text{range } T = V$

Rank-nullity thm: $\dim(\text{range } T) + \dim(\text{null } T) = \dim U$
 $\dim V + \dim\{\vec{0}\} = \dim U$
 $\rightarrow \dim V = \dim U$

(\Leftarrow) Let U, V be vector spaces of the same dimension. Let u_1, \dots, u_n be a basis for U and v_1, \dots, v_n be a basis for V .

Define the map $T: U \rightarrow V$ as follows:

$$T(c_1 u_1 + \dots + c_n u_n) = c_1 v_1 + \dots + c_n v_n$$

Since $v_i, i=1, \dots, n$ span V , T is surjective.

Since $v_i, i=1, \dots, n$ are linearly ind., $\text{null } T = \{\vec{0}\}$.

$\therefore T$ is injective and surjective $\Rightarrow T$ is an isomorphism

Linear maps and matrices

$$m \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix} \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix}_n = \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix}_m$$

$\nearrow \mathbb{C}^n / \mathbb{R}^n$
 $\nearrow \mathbb{C}^m / \mathbb{R}^m$

Example

Let $A \in M_{m \times n}$ be a fixed matrix. Then, we can define a linear map $T_A: \mathbb{F}^n \rightarrow \mathbb{F}^m$ via $T_A(\mathbf{v}) = A\mathbf{v}$, where we recall matrix vector multiplication $(A\mathbf{v})_i = \sum_{k=1}^n A_{ik}v_k$ for $i = 1, \dots, m$.

Next we will see that we can use matrices to represent linear maps between finite dimensional vector spaces.

$$\mathbb{P}_2: \text{basis } 1, x, x^2$$
$$5 - 10x + 45x^2 \Rightarrow \begin{pmatrix} 5 \\ -10 \\ 45 \end{pmatrix}$$

Definition

Let $T \in \mathcal{L}(U, V)$ where U and V are vector spaces. Let $\mathbf{u}_1, \dots, \mathbf{u}_n$ and $\mathbf{v}_1, \dots, \mathbf{v}_m$ be bases for U and V respectively. The matrix of T with respect to these bases is the $m \times n$ matrix $\mathcal{M}(T)$ with entries A_{ij} , $i = 1, \dots, m$, $j = 1, \dots, n$ defined by

$$T\mathbf{u}_k = A_{1k}\mathbf{v}_1 + \cdots + A_{mk}\mathbf{v}_m$$

i.e. the k th column of A is the scalars needed to write $T\mathbf{u}_k$ as a linear combination of the basis of V :

$$T\mathbf{u}_k = \sum_{i=1}^m A_{ik}\mathbf{v}_i$$

Note that since a linear map $T \in \mathcal{L}(U, V)$ is uniquely determined by its image on a basis of U , we see that once we pick basis of U and V its matrix representation is uniquely determined.

Example

Let $D \in \mathcal{L}(\mathbb{P}_4(\mathbb{R}), \mathbb{P}_3(\mathbb{R}))$ be the differentiation map, $Dp = p'$. Find the matrix of D with respect to the standard bases of $\mathbb{P}_3(\mathbb{R})$ and $\mathbb{P}_4(\mathbb{R})$.

Standard basis: $1, x, x^2, x^3, (x^4)$

$$T(u_1) = (1)' = 0$$

$$T(u_2) = (x)' = 1$$

$$T(u_3) = (x^2)' = 2x$$

$$T(u_4) = (x^3)' = 3x^2$$

$$T(u_5) = (x^4)' = 4x^3$$

The matrix is:

$$M(D) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

- Observe that if we choose bases $\mathbf{u}_1, \dots, \mathbf{u}_n$ and $\mathbf{v}_1, \dots, \mathbf{v}_m$ for U, V and represent $T \in \mathcal{L}(U, V)$ as a matrix $\mathcal{M}(T)$, then the corresponding map can be obtained by just working with the **coordinates of vectors in U, V** with respect to the chosen basis
- If $\mathbf{u} = \sum_{i=1}^n \alpha_i \mathbf{u}_i$, then the coordinates of $T(\mathbf{u})$ with respect to $\mathbf{v}_1, \dots, \mathbf{v}_m$ can be obtained by the matrix vector multiplication $\mathcal{M}(T)\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is the $n \times 1$ matrix with entries α_i

Example

If we want to find the derivative of $p = x^4 + 12x^3 - 5x^2 + 7$ with respect to the standard monomial basis of $\mathbb{P}_4(\mathbb{R})$, we use $\mathcal{M}(D)$ from the previous example to obtain

$$\mathcal{M}(D)\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 0 \\ -5 \\ 12 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -10 \\ 36 \\ 4 \end{pmatrix}.$$

Thus, translating back into the monomial basis of $\mathbb{P}_3(\mathbb{R})$ gives
 $D(p) = -10x + 36x^2 + 4x^3$.

$\frac{d}{dx}$

Other points

$$M(ST) = M(S)M(T)$$

- Looking at matrices as representations of linear maps gives us an intuitive explanation for why we do matrix multiplication the way we do! In fact, we want matrix multiplication to represent composition of linear maps
- We can use matrices to solve linear systems.

$$Ax = b$$

Determinants

Determinant

→ \mathbb{C} or \mathbb{R}

- The determinant is a function from $M_{n \times n} \rightarrow \mathbb{F}$, i.e. it is a function from the entries of a square matrix to a real or complex number.
- Notation: $\det(A) = |A|$
- The determinant has applications in solving linear systems, computing eigenvalues, etc

Example: 2×2 matrix

The determinant of a 2×2 matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example: 3×3 matrix

There is a **trick** for finding the determinant of a 3 by 3 matrix:

$$\begin{array}{ccc|ccc} a & b & c & a & b & \\ d & e & f & d & e & \\ g & h & i & g & h & \end{array}$$

$$|A| = aei + bfg + cdh - gec - hfa - idb$$

Cofactor expansion

For other $n \times n$ matrices, one can compute the determinant using **cofactor expansion**.

Definition (Cofactor expansion)

Let $A = \{a_{j,k}\}_{j,k=1}^n$ be a $n \times n$ matrix. Let $M_{j,k}$ denote the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing the j^{th} row and the k^{th} column of A . For each row $j = 1, \dots, n$

$$|A| = \sum_{k=1}^n a_{j,k} (-1)^{j+k} M_{j,k}.$$



Similarly, for each column $k = 1, \dots, n$

$$|A| = \sum_{j=1}^n a_{j,k} (-1)^{j+k} M_{j,k}.$$



The numbers $C_{j,k} = (-1)^{j+k} M_{j,k}$ are called *cofactors*.

Proposition

The determinant of a diagonal matrix or triangular matrix is the product of the entries on the diagonal.

Sketch of proof.

$$\begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}$$

$$\begin{aligned} |A| &= \alpha_1 \begin{vmatrix} \alpha_2 & 0 \\ 0 & \alpha_3 \end{vmatrix} (-1)^2 \\ &= \alpha_1 \alpha_2 \alpha_3 \end{aligned}$$

Inverse of a matrix

$$C_{j,k} = (-1)^{j+k} M_{j,k}$$

Theorem

Let A be an $n \times n$ invertible matrix and let $C = \{C_{j,k}\}_{j,k=1}^n$ be its cofactor matrix. Then

$$A^{-1} = \frac{1}{|A|} C^T$$

Note: The matrix A is invertible if and only if the linear map represented by the matrix is an isomorphism.

Cramer's rule

Corollary

Suppose A is an $n \times n$ invertible matrix. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution given by

$$x_i = \frac{|A_i|}{|A|}, \quad i = 1, \dots, n,$$

where A_i is the matrix obtained by replacing the i^{th} column of A with \mathbf{b} .

Transpose of a matrix

Definition

The *transpose* of an $m \times n$ matrix A is the $n \times m$ matrix, denoted A^T , defined entry-wise as $\{A_{j,k}^T\} = \{A_{k,j}\}$ for $j = 1, \dots, m$ and $k = 1, \dots, n$ (i.e. the rows of A are the columns of A^T and the columns of A are the rows of A^T)

Properties of the determinant

Proposition

$|A| \neq 0$ if and only if A is invertible

Proposition

Let A be an $n \times n$ real matrix.

- 1 If A has a zero column, then $|A| = 0$.
- 2 If A has two equal columns, then $|A| = 0$.
- 3 If one column of A is a multiple of another, then $|A| = 0$.
- 4 $|AB| = |A||B|$
- 5 $|\alpha A| = \alpha^n |A|$ for $\alpha \in \mathbb{F}$
- 6 $|A^T| = |A|$

Inner product spaces

Complex numbers

Recall that for a complex number $z = a + ib$, we define the following:

- Real part: $\operatorname{Re}(z) = a$,
- Imaginary part: $\operatorname{Im}(z) = b$,
- Complex conjugate: $\bar{z} = a - ib$,
- Modulus: $|z| = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2} = \sqrt{a^2 + b^2}$

We have $|z|^2 = z\bar{z}$ and $\operatorname{Re}(z) = \frac{z+\bar{z}}{2}$.

$$\begin{aligned} z\bar{z} &= (a+ib)(a-ib) \\ &= a^2 - i^2b^2 \\ &= a^2 + b^2 \end{aligned}$$

$$x \in \mathbb{R} : x = \overline{x}$$

Definition

Let V be an \mathbb{F} -vector space. A function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$ is called *inner product* on V if the following holds:

- 1 (Conjugate) symmetry: $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ for all $\mathbf{x}, \mathbf{y} \in V$, where \bar{a} denotes the complex conjugate for $a \in \mathbb{C}$
- 2 Linearity in the first argument: $\langle \alpha \mathbf{x} + \beta \mathbf{y}, \mathbf{z} \rangle = \alpha \langle \mathbf{x}, \mathbf{z} \rangle + \beta \langle \mathbf{y}, \mathbf{z} \rangle$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha, \beta \in \mathbb{F}$
- 3 Positive definiteness: $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$

A vector space equipped with an inner product is called an *inner product space*.

$$\mathbb{F} = \mathbb{C} : \langle \mathbf{x}, \alpha \mathbf{y} + \beta \mathbf{z} \rangle = \overline{\alpha} \langle \mathbf{x}, \mathbf{y} \rangle + \overline{\beta} \langle \mathbf{x}, \mathbf{z} \rangle$$

Example

- Standard inner product on \mathbb{R}^n : $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$
- Standard inner product on \mathbb{C}^n : $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i \bar{y}_i$ for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$
- On the space of polynomials $\mathbb{P}_n(\mathbb{R})$: $\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^1 p(x)q(x)dx$ for $\mathbf{p}, \mathbf{q} \in \mathbb{P}_n(\mathbb{R})$

Exercise: check conditions hold

Proposition

Let V be an inner product space. Then $\mathbf{x} = \mathbf{0}$ if and only if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for all $\mathbf{y} \in V$.

Proof.

(\Rightarrow) Suppose $\mathbf{x} = \mathbf{0}$. Note that $\mathbf{0}\mathbf{x} = \mathbf{x}$.

Then for $\mathbf{y} \in V$, $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{0}\mathbf{x}, \mathbf{y} \rangle = \mathbf{0}\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{0}$

(\Leftarrow) Suppose $\langle \mathbf{x}, \mathbf{y} \rangle = 0 \ \forall \mathbf{y} \in V$.

Then in particular it holds for $\mathbf{y} = \mathbf{x}$

$$\Rightarrow \langle \mathbf{x}, \mathbf{x} \rangle = 0 \quad \Rightarrow \quad \mathbf{x} = \mathbf{0}$$

Cauchy-Schwarz Inequality

Proposition

Let V be an inner product space. Then

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle}$$

for all $\mathbf{x}, \mathbf{y} \in V$.

Proof. Let $t \in \mathbb{F}$, $x, y \in V$.

$$\text{Then } 0 \leq \langle x - ty, x - ty \rangle = \langle x, x \rangle - t \langle y, x \rangle - \overline{t} \langle x, y \rangle + |t|^2 \langle y, y \rangle$$

This holds for any $t \in \mathbb{F}$. Choose $t = \frac{\langle x, y \rangle}{\langle y, y \rangle}$, $y \neq 0$

$$\Rightarrow 0 \leq \langle x, x \rangle - \frac{\langle x, y \rangle \langle y, x \rangle}{\langle y, y \rangle} - \frac{\langle y, x \rangle \langle x, y \rangle}{\langle y, y \rangle} + \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle}$$

$$= \langle x, x \rangle - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle}$$

$$\Rightarrow \langle y, y \rangle \langle x, x \rangle \geq |\langle x, y \rangle|^2$$

$$\text{Take } \mathcal{N} : \quad |\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle \langle y, y \rangle}$$

Proposition

Let V be an inner product space. Then $\langle \cdot, \cdot \rangle$ induces a norm on V via $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ for all $\mathbf{x} \in V$.

Proof.

References

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