Module 9: Linear Algebra III

Operational math bootcamp

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Outline

- Adjoints, unitaries and orthogonal matrices
- Orthogonal decomposition
- Spectral theory
  - Eigenvalues and eigenvectors
  - Algebraic and geometric multiplicity of eigenvalues
  - Matrix diagonalization

Recall

Definition

Let V be an $\mathbb{F}$-vector space. A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ is called *inner product* on V if the following holds:

1. (Conjugate) symmetry: $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ for all $\mathbf{x}, \mathbf{y} \in V$, where $\overline{a}$ denotes the complex conjugate for $a \in \mathbb{C}$
2. Linearity in the first argument: $\langle \alpha \mathbf{x} + \beta \mathbf{y}, \mathbf{z} \rangle = \alpha \langle \mathbf{x}, \mathbf{z} \rangle + \beta \langle \mathbf{y}, \mathbf{z} \rangle$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha, \beta \in \mathbb{F}$
3. Positive definiteness: $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$

A vector space equipped with an inner product is called an *inner product space*.

Recall

Inner product on $M_n(\mathbb{F})$: $\langle A, B\rangle = \mathrm{Tr}(B^*A)$

Example

- Standard inner product on $\mathbb{R}^n$: $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^n x_i y_i$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$
- Standard inner product on $\mathbb{C}^n$: $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^n x_i \overline{y}_i$ for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$
- On the space of polynomials $\mathbb{P}_n(\mathbb{R})$: $\langle \boldsymbol{p}, \boldsymbol{q}\rangle = \int_{-1}^1 p(x)q(x)\mathrm{d}x$ for $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}_n(\mathbb{R})$

Proposition

Cauchy-Schwarz Inequality

Let V be an inner product space. Then

$$|\langle \mathbf{x}, \mathbf{y}\rangle| \leq \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle}\sqrt{\langle \mathbf{y}, \mathbf{y}\rangle}$$

for all $\mathbf{x}, \mathbf{y} \in V$.

Proposition

Let V be an inner product space. Then $\langle \cdot, \cdot \rangle$ induces a norm on V via $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ for all $\mathbf{x} \in V$.

Proof.

(1) Norm: $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$

We have $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \iff x = 0$

Prop 3 of IP gives us this property.

(2) $\|\alpha x\| = |\alpha| \|x\|$

IP: $\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x\rangle} = \sqrt{\alpha\bar{\alpha}\langle x, x\rangle}$

$= |\alpha|\sqrt{\langle x, x\rangle}$

(3) Δ inequality $\|x+y\| \leq \|x\| + \|y\|$

For $x, y \in V$, $\|x+y\|^2 = \langle x+y, x+y\rangle$

$= \langle x, x\rangle + \langle y, x\rangle + \langle x, y\rangle$

$+ \langle y, y\rangle$

$= \|x\|^2 + 2\,\mathrm{Re}\langle x, y\rangle + \|y\|^2$

$$\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2$$
$$\leq (\|x\| + \|y\|)^2$$

Note: With this identification the Cauchy-Schwarz inequality can be restated as: $|\langle \mathbf{x}, \mathbf{y}\rangle| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in V$.

Example

The norm introduced by the standard inner product on $\mathbb{R}^n$ is the Euclidean distance.

Adjoint

Definition

Let U, V be inner product spaces and $S \colon U \to V$ be a linear map. The *adjoint* S^* of S is the linear map $S^* \colon V \to U$ defined such that

$$\boxed{\langle S\mathbf{u}, \mathbf{v}\rangle_V = \langle \mathbf{u}, S^*\mathbf{v}\rangle_U} \qquad \text{for all } \mathbf{u} \in U, \mathbf{v} \in V.$$

Proposition

Let U, V be inner product spaces and $S: U \to V$ be a linear map. Then S^* is unique and linear.

Proof.

Uniqueness: Suppose $\exists\, T, R : V \to U$ s.t. $\forall u \in U, \forall v \in V$

$$\langle Su, v \rangle = \langle u, Tv \rangle = \langle u, Rv \rangle$$

$$\Rightarrow \overline{\langle Tv, u \rangle} = \overline{\langle Rv, u \rangle}$$

$$\Rightarrow Tv = Rv \quad \forall v \in V$$

$$\Rightarrow T = R$$

Proposition

Let U, V be inner product spaces and $S\colon U \to V$ be a linear map. Then S^* is unique and linear.

Proof. S^* is linear

$\alpha \in \mathbb{F}$, $v, w \in V$, $u \in U$

$$\langle u, S^*(\alpha v + w)\rangle = \langle Su, \alpha v + w\rangle$$
$$= \bar{\alpha}\langle Su, v\rangle + \langle Su, w\rangle$$
$$= \bar{\alpha}\langle u, S^* v\rangle + \langle u, S^* w\rangle$$
$$= \langle u, \alpha S^* v + S^* w\rangle$$

$$S^*(\alpha v + w) = \alpha S^* v + S^* w$$

Example

Define $S\colon \mathbb{R}^3 \to \mathbb{R}^2$ by $S\mathbf{x} = (2x_1 + x_3, -x_2)$. What is the adjoint operator S^*?

$\langle Sx, y\rangle_{\mathbb{R}^2} = \langle x, S^* y\rangle_{\mathbb{R}^3}$. Find S^*.

Let $x \in \mathbb{R}^3$, $y \in \mathbb{R}^2$.

$$\langle Sx, y\rangle = \langle (2x_1 + x_3, -x_2), (y_1, y_2)\rangle$$
$$= y_1(2x_1 + x_3) - x_2 y_2$$
$$= 2y_1 x_1 + y_1 x_3 - y_2 x_2$$
$$= \langle (x_1, x_2, x_3), (2y_1, -y_2, y_1)\rangle_{\mathbb{R}^3}$$

$$:= \langle x, S^* y \rangle_{\mathbb{R}^3}$$

$$S^* y = (2y_1, -y_2, y_1)$$

Proposition

Let $A \in M_{m \times n}(\mathbb{F})$ be a matrix and $T_A : \mathbb{F}^n \to \mathbb{F}^m : \mathbf{x} \mapsto A\mathbf{x}$. Then, $T_A^*(\mathbf{x}) = A^*\mathbf{x}$, where $A^* \in M_{n \times m}(\mathbb{F})$ with $(A^*)_{ij} = \overline{A_{ji}}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

In particular, if $\mathbb{F} = \mathbb{R}$, the adjoint of the matrix is given by its transpose,denoted A^T, and if $\mathbb{F} = \mathbb{C}$, it is given by its conjugate transpose, denoted A^*.

$$\mathbb{R}: \langle Ax, y \rangle = \langle x, A^T y \rangle$$

$$\mathbb{C}: \langle Bx, y \rangle = \langle x, B^* y \rangle$$

Proof for $\mathbb{R}$:

$$\langle Ax, y\rangle_{\mathbb{R}^m} = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} A_{ji}x_i\right)y_j$$

$$= \sum_{i=1}^{n} x_i \underbrace{\left(\sum_{j=1}^{m} A_{ji}\, y_j\right)}_{?\,=\,A^T y}$$

$$= \langle x, A^T y\rangle_{\mathbb{R}^n}$$

Definition

A matrix $O \in M_n(\mathbb{R})$ is called *orthogonal* if its inverse is given by its transpose, i.e. $O^T O = OO^T = I$.

A matrix $U \in M_n(\mathbb{C})$ is called *unitary* if the inverse is given by the conjugate transpose, i.e. $U^* U = UU^* = I$.

$U \in M_n(\mathbb{R})$:

$$\langle Ux, Uy \rangle = \langle x, U^T U y \rangle = \langle x, y \rangle$$

$U \in M_n(\mathbb{C})$

$$= \langle x, U^* U y \rangle = \langle x, y \rangle$$

Example

- Let $\varphi \in [0, 2\pi]$. Then

$$\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}$$

  is an orthogonal matrix. What does it describe geometrically?

- The following is a unitary matrix:

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Definition

Let $A \in M_n(\mathbb{F})$. We call A *self-adjoint* if $A^* = A$. In the case $\mathbb{F} = \mathbb{R}$, such an A is called *symmetric* and if $\mathbb{F} = \mathbb{C}$, such an A is called *Hermitian*.

$A, B \in M_n(\mathbb{F})$

Check that $\langle A, B \rangle = \mathrm{Tr}(B^* A)$ is an IP, where $\mathrm{Tr}(A) = \sum_{i=1}^{n} A_{ii}$

Orthogonality and Gram-Schmidt

Definition

Two vectors $\mathbf{x}, \mathbf{y} \in V$ are called *orthogonal* if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, denoted $\mathbf{x} \perp \mathbf{y}$. We call them *orthonormal* if additionally the vectors are normalized, i.e. $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$. A basis $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of V is called *orthonormal basis (ONB)*, if the vectors are pairwise orthogonal and normalized.

Proposition

Let $\mathbf{x}_1, \ldots, \mathbf{x}_k \in V$ be orthonormal. Then the system of vectors is linearly independent.

Proof. Show $0 = \sum_{i=1}^{k} \alpha_i x_i \Rightarrow \alpha_i = 0 \quad \forall i = 1, \ldots, k$

Let $0 = \sum_{i=1}^{k} \alpha_i x_i \Longrightarrow \left\| \sum_{i=1}^{k} \alpha_i x_i \right\|^2 = 0$

$$0 = \left\| \sum_{i=1}^{k} \alpha_i x_i \right\|^2 = \left\langle \sum_{i=1}^{k} \alpha_i x_i , \sum_{j=1}^{k} \alpha_j x_j \right\rangle$$

$$= \sum_{i,j} \langle \alpha_i x_i , \alpha_j x_j \rangle$$

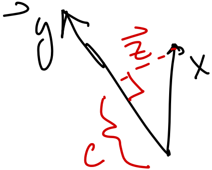
$$= \sum_{i=1}^{k} |\alpha_i|^2 \underbrace{\langle x_i, x_i \rangle}_{=1} + \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \alpha_i \overline{\alpha_j} \underbrace{\langle x_i, x_j \rangle}_{=0}$$

$$\Rightarrow 0 = \sum_{i=1}^{n} |\alpha_i|^2 \Rightarrow \alpha_i = 0 \quad \forall i = 1, \dots, k$$

Proposition (Orthogonal Decomposition)

Let $\mathbf{x}, \mathbf{y} \in V$ with $\mathbf{y} \neq 0$. Then, there exist $c \in F$ and $\mathbf{z} \in V$ such that $\mathbf{x} = c\mathbf{y} + \mathbf{z}$ with $\mathbf{y} \perp \mathbf{z}$.

Given a basis, we can obtain an ONB from it using the Gram-Schmidt algorithm by repeating this orthogonal decomposition.



Proposition (Gram-Schmidt Algorithm)

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in V$ be a system of linearly independent vectors. Define $\mathbf{y}_1 = \mathbf{x}_1 / \|\mathbf{x}_1\|$. For $i = 2, \ldots, n$ define $\mathbf{y}_j$ inductively by

$$\mathbf{y}_i = \frac{\mathbf{x}_i - \sum_{k=1}^{i-1} \langle \mathbf{x}_i, \mathbf{y}_k \rangle \mathbf{y}_k}{\|\mathbf{x}_i - \sum_{k=1}^{i-1} \langle \mathbf{x}_i, \mathbf{y}_k \rangle \mathbf{y}_k\|}.$$

Then the $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are orthonormal and

$$\operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_n\} = \operatorname{span}\{\mathbf{y}_1, \ldots, \mathbf{y}_n\}.$$

The proof is omitted but can be found in the book.

Recall: connection between matrices and linear maps

Multiplication by a matrix defines a linear map

Let $A \in M_{m\times n}$ be a fixed matrix. Then, we can define a linear map $T_A \colon \mathbb{F}^n \to \mathbb{F}^m$ via $T_A(\mathbf{v}) = A\mathbf{v}$, where we recall matrix vector multiplication $(A\mathbf{v})_i = \sum_{k=1}^{n} A_{ik} v_k$ for $i = 1, \ldots, m$.

Given a bases for U and V, $T : U \to V$ can be written as a matrix

Let $T \in \mathcal{L}(U, V)$ where U and V are vector spaces. Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ and $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be bases for U and V respectively. The matrix of T with respect to these bases is the $m \times n$ matrix $\mathcal{M}(T)$ with entries A_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$ defined by

$$T\mathbf{u}_k = A_{1k}\mathbf{v}_1 + \cdots + A_{mk}\mathbf{v}_m.$$

Eigenvalues

Definition

Given an operator $A\colon V \to V$ and $\lambda \in \mathbb{F}$, λ is called an *eigenvalue* of A if there exists a non-zero vector $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

We call such $\mathbf{v}$ an *eigenvector* of A with eigenvalue λ. We call the set of all eigenvalues of A *spectrum* of T and denote it by $\sigma(T)$.

Motivation in terms of linear maps: Let $T\colon V \to V$ be a linear map, where V is a vector space. We would like to describe the action of this linear map in a particularly "nice" way: such that T acts only by scaling, i.e. $T\mathbf{v}_i = \lambda_i\mathbf{v}_i$ where $\lambda_i \in \mathbb{F}$ for $i = 1, \ldots, n$.

Finding eigenvalues

Note: here we will assume $\mathbb{F} = \mathbb{C}$, so that we are working on an algebraically closed field.

- Rewrite $A\mathbf{v} = \lambda\mathbf{v}$ as $(A - \lambda I)v = 0$
- Thus, if λ is an eigenvalue, we can find the corresponding eigenvectors by finding the null space of $A - \lambda I$.
- The subspace null$(A - \lambda I)$ is called the *eigenspace*
- To find the eigenvalues of A, one must find the scalars λ such that null$(A - \lambda I)$ contains non-trivial vectors (i.e. not $\mathbf{0}$)
- Recall: We saw that $T \in \mathcal{L}(U, V)$ is injective if and only if null $T = \{\mathbf{0}\}$.
- Thus λ is an eigenvalue if and only if $A - \lambda I$ is not invertible.
- Recall: $|A| \neq 0$ if and only if A is invertible.
- Thus λ is an eigenvalue if and only if $|A - \lambda I| = 0$

Theorem

The following are equivalent

1. $\lambda \in \mathbb{F}$ *is an eigenvalue of* A*,*
2. $(A - \lambda I)\mathbf{v} = 0$ *has a non-trivial solution,*
3. $|A - \lambda I| = 0$*.*

Characteristic polynomial

Definition

If A is an $n \times n$ matrix, $p_A(\lambda) = |A - \lambda I|$ is a polynomial of degree n called the *characteristic polynomial* of A.

To find the eigenvectors of A, one needs to find the roots of the characteristic polynomial.

$$p_A(\lambda) = |A - \lambda I| = 0$$

Example

Find the eigenvalues of

$$A = \begin{bmatrix} 4 & -2 \\ 5 & -3 \end{bmatrix}.$$

$$0 = |A - \lambda I|$$

$$= \begin{vmatrix} 4-\lambda & -2 \\ 5 & -3-\lambda \end{vmatrix}$$

$$= -(3+\lambda)(4-\lambda) + 10$$

$$= \lambda^2 - \lambda - 2$$

$$= (\lambda - 2)(\lambda + 1) \qquad \therefore \lambda = -1, +2$$

Multiplicity

Example: $p(\lambda) = (\lambda - 1)^2(\lambda - 2)(\lambda + 3)^4$

Definition

The multiplicity of the root λ in the characteristic polynomial is called the *algebraic multiplicity* of the eigenvalue λ. The dimension of the eigenspace $\text{null}(A - \lambda I)$ is called the *geometric multiplicity* of the eigenvalue λ.

Definition (Similar matrices)

Square matrices A and B are called *similar* if there exists an invertible matrix S such that

$$A = SBS^{-1}.$$

Similar matrices have the same characteristic polynomials and hence the same eigenvalues (see exercise).

Theorem

Suppose A is a square matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be eigenvectors corresponding to these eigenvalues. Then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

Proof. By induction on n:

Base case: $n = 1$. This is trivial, one non-zero vector is ~~always~~ is always linearly independent.

Inductive hypothesis: Suppose the claim holds for $k \geq 1$. In particular, $v_1, \ldots, v_k$ are linearly independent.

Suppose λ_{k+1} is an eigenvalue for A with corresponding eigenvector v_{k+1} and $\lambda_{k+1} \neq \lambda_k \neq \cdots \neq \lambda_1$.

Let $0 = \sum_{i=1}^{k+1} \alpha_i v_i$, $\alpha_i \in \mathbb{F}$

Apply $(A - \lambda_{k+1} I)$ to both sides:

$$0 = \sum_{i=1}^{k+1} \alpha_i (A - \lambda_{k+1} I) v_i$$

$$= \sum_{i=1}^{k+1} \alpha_i A v_i - \sum_{i=1}^{k+1} \alpha_i \lambda_{k+1} I v_i$$

$$= \sum_{i=1}^{k+1} \alpha_i (\lambda_i v_i - \lambda_{k+1} v_i)$$

$$= \sum_{i=1}^{k} \alpha_i \underbrace{(\lambda_i - \lambda_{k+1})}_{\neq 0} \underbrace{v_i}_{\neq 0} + \alpha_{k+1} \underbrace{(\lambda_{k+1} - \lambda_{k+1})}_{=0} \underbrace{v_{k+1}}_{\neq 0}$$

Corollary

If a $A \in M_n(\mathbb{C})$ has n distinct eigenvalues, then A is diagonalizable. That is there exists an invertible matrix $S \in M_n(\mathbb{C})$ such that $A = SDS^{-1}$, where D is a diagonal matrix with the eigenvalues of A in the diagonal.

$$0 = \sum_{i=1}^{k} \alpha_i \underbrace{(\lambda_i - \lambda_{k+1}) v_i}_{\neq 0} \Rightarrow \alpha_i = 0 \ \forall i = 1, \dots, k$$

$$\therefore 0 = \alpha_{k+1} v_{k+1} \Rightarrow \alpha_{k+1} = 0$$

Theorem

Let $A : V \to V$ be an operator with n eigenvalues. A is diagonalizable if and only if for each eigenvalue λ, the geometric multiplicity of λ and the algebraic multiplicity of λ are the same.

Example: a diagonalizable matrix

$$A = \begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix} \text{ is diagonalizable.}$$

Find eigenvalues:

$$0 = |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 8 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 16$$

$$= \lambda^2 - 2\lambda - 15$$

$$= (\lambda - 5)(\lambda + 3)$$

$$\therefore \lambda = -3, 5$$

Find eigenvectors:

Example continued

$$A+3I = \begin{pmatrix} 4 & 2 \\ 8 & 4 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(1,-2)$ spans $\text{null}(A+3I)$

$$A-5I = \begin{pmatrix} -4 & 2 \\ 8 & -4 \end{pmatrix} \sim \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow (1,2) \text{ spans } \text{null}(A-5I)$$

Example continued

$$A = \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}^{-1}}_{S^{-1}}$$

Example: a matrix that is not diagonalizable

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ is } \textit{not} \text{ diagonalizable.}$$

$$0 = |B - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$$

$\lambda = 1$, w/ algebraic multiplicity 2

$$B - I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \text{null}(B-I) \text{ is}$$

spanned by $(1, 0)$

$\therefore \lambda = 1$ has geometric mult 1

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