Exercises for Module 10: Differentiation and Integration

1. Show that

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

is smooth.

Clearly f is smooth at all $x \neq 0$. Thus, we only need to look at the behaviour of f at 0. Since $f^{(k)}(x) = 0 \ \forall x \in [-\infty, 0], \forall k \geq 0$, we need to show that $\lim_{h \downarrow 0} \frac{f^{(k)}(h) - f^{(k)}(0)}{h} = \lim_{h \to 0} \frac{f^{k}(h)}{h} = 0 \ \forall k \geq 0$.

This is true for $k=0$ since $\lim_{h \downarrow 0} \frac{e^{-1/h}}{h} = \lim_{h \downarrow 0} \frac{\left(\frac{1}{h}\right)}{e^{1/h}} \overset{H}{=} \lim_{h \downarrow 0} \frac{\left(\frac{-1}{h^2}\right)}{\frac{-1}{h^2} e^{1/h}} = \lim_{h \downarrow 0} e^{-1/h} = 0$.

First we prove the following using induction: $f^{(k)}(x) = p_{2k}(x^{-1}) x^{-1/x}$ where p_{2k} is a polynomial of degree $2k$.

Base case: $f'(x) = \frac{1}{x^2} e^{-1/x} = (x^{-1})^2 e^{-1/x}$ as required

Inductive hypothesis: $f^{(m)}(x) = p_{2m}(x^{-1}) e^{-1/x}$ for some $m \geq 1$.

Then $f^{(m+1)}(x) = (p_{2m}(x^{-1}))' e^{-1/x} + \frac{1}{x^2} p_{2m}(x^{-1}) e^{-1/x}$

$= p_{2m+1}(x^{-1}) e^{-1/x} + p_{2m+2}(x^{-1}) e^{-1/x} = p_{2(m+1)}(x^{-1}) e^{-1/x}$.

Thus $f^{(k)}(x) = p_{2k}(x^{-1}) e^{-1/x} \quad \forall k \geq 1$.

Finally, since $\lim_{h \downarrow 0} \frac{f^{(k)}(h)}{h} = \lim_{h \downarrow 0} p_{2k+1}(h^{-1}) e^{-1/h} = \lim_{h \downarrow 0} \frac{p_{2k+1}(h^{-1})}{e^{1/h}} \overset{H}{=} \lim_{h \downarrow 0} \frac{p_{2k+2}(h^{-1})}{p_2(h^{-1}) e^{1/h}} = \cdots = 0$ by repeated applications of l'Hôpital's rule.

2. Let $f \in \mathcal{R}([a,b])$ and suppose $|f| \leq M$ for some $M > 0$. Show that $|\int_a^b f(x)dx| \leq M(b-a)$.

Proof. By definition, $-|f(x)| \leq f(x) \leq |f(x)| \quad \forall x \in [a,b]$.

By monotonicity, $-\int_a^b |f(x)| dx \leq \int_a^b f(x) dx \leq \int_a^b |f(x)| dx$

$\Rightarrow \left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx \ \forall x \in [a,b]$ by def of abs value

$\leq \int_a^b M \, dx$ by monotoncity

$= M(b-a)$ by integral of a constant

Note that in this proof we have shown that for $f \in \mathcal{R}([a,b])$,

$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$. ⊛

3. Prove the Higher-Order Leibniz product rule, i.e. for $f, g \in C^r([a,b])$ we have

$$(fg)^{(r)}(x) = \sum_{k=0}^{r} \binom{r}{k} f^{(k)}(x) g^{(r-k)}(x).$$

You can use properties of the binomial coefficient.

We prove this by induction on r.

Base case $r=1$

$$(fg)^{(1)}(x) = (fg)'(x) = f'(x)g(x) + f(x)g'(x) \quad \text{by regular product rule}$$

$$= \sum_{k=0}^{1} \binom{1}{k} f^{(k)}(x) g^{(1-k)}(x)$$

Inductive hypothesis: suppose for some $n \geq 1$,

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x).$$

We show the statement holds for $n+1$:

$$(fg)^{(n+1)}(x) = \frac{d}{dx}\left((fg)^{(n)}(x)\right) = \frac{d}{dx} \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x) \quad \text{by inductive hypothesis}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left[(f^{(k)})'(x) g^{(n-k)}(x) + f^{(k)}(x) (g^{(n-k)})'(x) \right] \quad \text{by product rule}$$

$$= \sum_{k=0}^{n} \binom{n}{k} f^{(k+1)}(x) g^{(n-k)}(x) + \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k+1)}(x)$$

$\hat{k} = k+1$

$$= \sum_{\hat{k}=1}^{n+1} \binom{n}{\hat{k}-1} f^{(\hat{k})}(x) g^{(n-\hat{k}+1)}(x) + \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k+1)}(x)$$

$$= \binom{n}{n} f^{(n+1)}(x) g(x) + \sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k} \right] f^{(k)}(x) g^{(n-k+1)}(x) + \binom{n}{0} f(x) g^{(n+1)}(x)$$

Note that $\binom{n}{k-1} + \binom{n}{k}$

$$= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

$$= \frac{n! \cdot k + n! \cdot (n-k+1)}{k!(n-k+1)!}$$

$$= \frac{n!(n+1)}{k!(n+1-k)!}$$

$$= \binom{n+1}{k}$$

$$= \overbrace{f^{(n+1)}(x) g(x)}^{(n+1)^{st} \text{ term}} + \underbrace{\sum_{k=1}^{n} \binom{n+1}{k} f^{(k)}(x) g^{(n+1-k)}(x)}_{\text{middle terms}} + \underbrace{f(x) g^{(n+1)}(x)}_{\to 0^{th} \text{ term}}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}(x) g^{(n+1-k)}(x) \quad \text{as required.}$$

4. (Challenge Problem) Consider the space of continuous functions on the unit interval, $C([0,1])$. Prove that there exists a unique $f \in C([0,1])$ such that for all $x \in [0,1]$

$$f(x) = x + \int_0^x s f(s) \mathrm{d}s.$$

Hint: You can use that $C([0,1])$ is a complete metric space with respect to the supremums metric $d_\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$ for $f, g \in C([0,1])$.

We use the Banach fixed point theorem. We need to show that the map $G: C([0,1]) \to C([0,1])$ defined by

$$G(f)(x) = f(x) + \int_0^x s f(s)\, ds$$ is a contraction.

Note that G is a continuous function on $[0,1]$.

Using the sup norm, we need to show that $d_\infty(G(f_1), G(f_2)) \leq k\, d_\infty(f_1, f_2)$ for some $k<1$.

Let $f_1, f_2 \in C([0,1])$. Then

$$\begin{aligned}
d_\infty(G(f_1), G(f_2)) &= \sup_{x\in[0,1]} |G(f_1)(x) - G(f_2)(x)| \\
&= \sup_{x\in[0,1]} \left| \cancel{x} + \int_0^x s f_1(s)\, ds - \cancel{x} - \int_0^x s f_2(s)\, ds \right| \\
&= \sup_{x\in[0,1]} \left| \int_0^x s(f_1(s) - f_2(s))\, ds \right| \\
&\leq \sup_{x\in[0,1]} \int_0^x |s(f_1(s) - f_2(s))|\, ds \quad \text{by } (*) \text{ from exercise 2} \\
&= \sup_{x\in[0,1]} \int_0^x s\, |f_1(s) - f_2(s)|\, ds \\
&\leq \sup_{x\in[0,1]} \int_0^x s\, d_\infty(f_1, f_2)\, ds \quad \text{since } |f_1(s) - f_2(s)| \leq \sup_{s\in[0,1]} |f_1(s) - f_2(s)| \\
&= \sup_{x\in[0,1]} d_\infty(f_1, f_2) \int_0^x s\, ds \quad \text{by linearity of integral} \\
&= \sup_{x\in[0,1]} d_\infty(f_1, f_2) \frac{x^2}{2} \\
&= \frac{1}{2} d_\infty(f_1, f_2) \quad \therefore G \text{ is a contraction}
\end{aligned}$$

Since $G: C([0,1]) \to C([0,1])$ is a contraction and $C([0,1])$ is a complete metric space, such a unique f must exist by the Banach fixed pt Theorem.