Exercises for Module 2: Set Theory

1 Let $A = \{x \in \mathbb{R} : x < 100\}$, $B = \{x \in \mathbb{Z} : |x| \geq 20\}$, and $C = \{y \in \mathbb{N} : y \text{ is prime}\}$ ($A, B, C \subseteq \mathbb{R}$). Find $A \cap B$, $B^c \cap C$, $B \cup C$, and $(A \cup B)^c$.

$A \cap B = \{x \in \mathbb{Z} : (x \leq -20) \wedge (20 \leq x < 100)\}$

$B^c = (-20, 20) \cup \{x \in \mathbb{R} \setminus \mathbb{Z} : |x| \geq 20\}$

$B^c \cap C = \{y \in \mathbb{N} : y \text{ is prime and } y < 20\}$

$B \cup C = \{x \in \mathbb{Z} : |x| \geq 20\} \cup \{y \in \mathbb{N} : y \text{ is prime and } y < 20\}$

$A \cup B = \{x \in \mathbb{R} : (x < 100) \vee (x \in \mathbb{Z} \wedge x \geq 100)\}$

$(A \cup B)^c = \{x \in \mathbb{R} \setminus \mathbb{Z} : x > 100\}$

2. Is $\mathbb{R} \times \mathbb{R}$ with the ordering $(x_1, y_1) \preceq (x_2, y_2)$ if $x_1 \leq x_2$ a partially ordered set?

No. Take $(x_1, y_1) = (2, -1)$ and $(x_2, y_2) = (2, 8)$.

Then we have $(2, -1) \preceq (2, 8)$ and $(2, 8) \preceq (2, -1)$

but $(2, -1) \neq (2, 8)$.

The ordering is not anti-symmetric $\Rightarrow$ not a partial order.

3. Let S be a non-empty set. A relation R on S is called an equivalence relation if it is

(i) Reflexive: $(x, x) \in R$ for all $x \in S$

(ii) Symmetric: if $(x, y) \in R$ then $(y, x) \in R$ for all $x, y \in S$

(iii) Transitive: if $(x, y), (y, z) \in R$ then $(x, z) \in R$ for all $x, y, z \in S$

Given $x \in S$, the equivalence class of x (with respect to a given equivalence relation R) is defined to consist of those $y \in S$ for which $(x, y) \in R$. Show that two equivalence classes are either disjoint or identical.

Let $x_1, x_2 \in S$ such that $x_1 \neq x_2$. Let E_1 be the equivalence class of x_1 and E_2 be the equivalence class of x_2.

Note that any two sets are either disjoint or not disjoint. If E_1 & E_2 are disjoint, we are done. So we assume E_1 and E_2 are not disjoint, with the goal of showing that they are identical.

Since we assumed E_1, E_2 are not disjoint, $\exists y \in S$ such that $y \in E_1$ and $y \in E_2$, i.e., $(x_1, y) \in R$ and $(x_2, y) \in R$.

Let $z \in E_1 \iff (x_1, z) \in R$ by definition.

$\iff (y, z) \in R$ by symmetry & transitivity $((x_1, y) \in R)$

$\iff (x_2, z) \in R$ by symmetry & transitivity $((x_2, y) \in R)$

$\iff z \in E_2$.

Therefore $E_1 = E_2$.

4. Let $(X, \leq)$ be a partially ordered set and $S \subseteq X$ be bounded. Show that the infimum and supremum of S are unique (if they exist).

Proof that the supremum is unique (infimum is similar):

Let $(X, \leq)$ be a partially ordered set and $S \subseteq X$ bounded. Suppose in order to derive a contradiction that S has two suprema: r_1 and r_2.

By the definition of supremum, since r_1 is the sup and r_2 is another upper bound, $r_1 \leq r_2$.

Similarly, since r_2 is the sup and r_1 is an upper bound, $r_2 \leq r_1$.

By anti-symmetry of the partial order $\leq$, $r_1 = r_2$.

Thus if a partially ordered set has a supremum, it must be unique.

5. Let $S, T \subseteq \mathbb{R}$ and suppose both are bounded above. Define $S+T = \{s+t : s \in S, t \in T\}$. Show that $S+T$ is bounded above and $\sup(S+T) = \sup S + \sup T$.

Proof. Since both S & T are bounded above, they both have a supremum. Let $x = \sup S$ and $y = \sup T$. By definition, $s \le x$ $\forall s \in S$ and $t \le y$ $\forall t \in T$. Therefore $s+t \le x+y$ $\forall s \in S$, $\forall t \in T$, so $s+t$ is an upper bound for $S+T$ (i.e., $S+T$ is bounded above).

Claim: $\sup(S+T) = x+y$. We use the characterization of sup from Prop 2.22. We have already shown that $s+t$ is an upper bound for $S+T$, so it remains to show that $\forall \varepsilon > 0$ $\exists s+t \in (S+T)$ such that $x+y-\varepsilon < s+t$.

Let $\varepsilon > 0$ be arbitrary.

Since $x = \sup S$, $\exists s \in S$ such that $x - \varepsilon/2 < s$. ①

Since $y = \sup T$, $\exists t \in T$ such that $y - \varepsilon/2 < t$. ②

Thus $\exists s \in S, \exists t \in T$ such that $\underbrace{x+y-\varepsilon < s+t}_{①+②}$. $\therefore \sup(S+T) = \sup(S) + \sup(T)$

6. Let $f : X \to Y$, $X, Y \subseteq \mathbb{R}$, be defined by the map $x \mapsto \sin(x)$. For what choices of X and Y is f injective, surjective, bijective, or neither?

Solution is not unique. Here is one solution.

injective: $X = [-\frac{\pi}{2}, \frac{\pi}{2}]$, $Y = \mathbb{R}$

surjective: $X = \mathbb{R}$, $Y = [-1, 1]$

bijective: $X = [-\frac{\pi}{2}, \frac{\pi}{2}]$, $Y = [-1, 1]$

neither: $X = \mathbb{R}$, $Y = \mathbb{R}$