Module 2: Set Theory

Operational math bootcamp

Statistical Sciences
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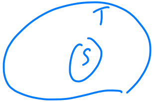
Outline

- Review of basic set theory
- Ordered Sets
- Functions

Introduction to Set Theory

- We define a *set* to be a collection of mathematical objects.
- If S is a set and x is one of the objects in the set, we say x is an element of S and denote it by $x \in S$.
- The set of no elements is called empty set and is denoted by $\emptyset$.

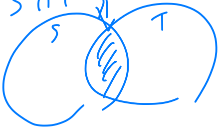


Definition (Subsets, Union, Intersection)

Let S, T be sets.

- We say that S is a *subset* of T, denoted $S \subseteq T$, if $s \in S$ implies $s \in T$.
- We say that $S = T$ if $S \subseteq T$ and $T \subseteq S$.
- We define the *union* of S and T, denoted $S \cup T$, as all the elements that are in *either* S or T.
- We define the *intersection* of S and T, denoted $S \cap T$, as all the elements that are in *both* S and T.
- We say that S and T are *disjoint* if $S \cap T = \emptyset$.

$S \cap T$?



$S \cup T$

Some examples

integers

real

Example

$\mathbb{N} \subseteq \mathbb{N}_0 \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ complex

natural $\quad$ $= \mathbb{N} \cup \{0\}$ $\quad$ rational

Example

Let $a, b \in \mathbb{R}$ such that $a < b$.
Open interval: $(a, b) := \{x \in \mathbb{R} : a < x < b\}$ (a, b may be $-\infty$ or $+\infty$)
Closed interval: $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$
We can also define half-open intervals.

$(a, b]$, $[a, b)$

Example

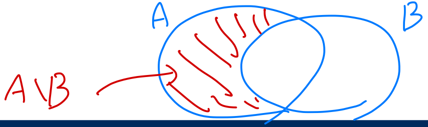
Let $A = \{x \in \mathbb{N} : 3|x\}$ and $B = \{x \in \mathbb{N} : 6|x\}$ Show that $B \subseteq A$.

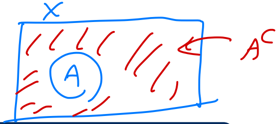
Proof. Let $x \in B$.

By the definition of B, $\exists\, k \in \mathbb{N}$, s.t. $x = 6k$.

Then $x = 3\underbrace{(2k)}_{\in\, \mathbb{N}} \in A$.

Difference of sets





Definition

Let $A, B \subseteq X$. We define the *set-theoretic difference* of A and B, denoted $A \setminus B$ (sometimes $A - B$) as the elements of X that are in A but *not* in B.
The complement of a set $A \subseteq X$ is the set $A^c := X \setminus A$.

Example

Let $X \subseteq \mathbb{R}$ be defined as $X = \{x \in \mathbb{R} : 0 < x \leq 40\} = (0, 40]$. Then

$X^c = (-\infty, 0] \cup (40, \infty)$

Recall that for sets S, T:

- the *union* of S and T, denoted $S \cup T$, is all the elements that are in *either* S and T
- and the *intersection* of S and T, denoted $S \cap T$, is all the elements that are in *both* S and T.

We extend the definition of union and intersection to an arbitrary family of sets as follows:

Definition

Let S_α, $\alpha \in A$, be a family of sets. A is called the *index set*. We define

$$\bigcup_{\alpha \in A} S_\alpha := \{x : \exists \alpha \text{ such that } x \in S_\alpha\},$$

$$\bigcap_{\alpha \in A} S_\alpha := \{x : x \in S_\alpha \text{ for all } \alpha \in A\}.$$

Example

$$\bigcup_{n=1}^{\infty}[-n, n] = \mathbb{R}$$

$$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

Theorem (De Morgan's Laws)

Let $\{S_\alpha\}_{\alpha \in A}$ *be an arbitrary collection of sets. Then*

$$\underbrace{\left(\bigcup_{\alpha\in A} S_\alpha\right)^c = \bigcap_{\alpha\in A} S_\alpha^c}\quad \text{and} \quad \left(\bigcap_{\alpha\in A} S_\alpha\right)^c = \bigcup_{\alpha\in A} S_\alpha^c$$

Proof.

$$\bigcup_{\alpha\in A} S_\alpha = \{x : \underbrace{\exists \alpha \text{ s.t. } x \in S_\alpha}_{\text{negate.}}\}.$$

$$\left(\bigcup_{\alpha\in A} S_\alpha\right)^c = \{x : \forall \alpha \in A,\ x \in S_\alpha^c\} = \bigcap_{\alpha\in A} S_\alpha^c.$$

To prove the second result, let $T_\alpha = S_\alpha^c$.

Then, by the first result,

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$$\left(\bigcup_{\alpha \in A} T_\alpha \right)^c = \bigcap_{\alpha \in A} T_\alpha^c .$$

Taking the complement of both sides,

$$\bigcup_{\alpha \in A} T_\alpha = \left(\bigcap_{\alpha \in A} T_\alpha^c \right)^c$$

Substitute $T_\alpha = S_\alpha^c$

$$\bigcup_{\alpha \in A} S_\alpha^c = \left(\bigcap_{\alpha \in A} S_\alpha \right)^c .$$

Since a set is itself a mathematical object, a set can itself contain sets.

Definition

The power set $\mathcal{P}(S)$ of a set S is the set of all subsets of S.

Example

Let $S = \{a, b, c\}$.
Then $\mathcal{P}(S) =$ $\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, S.$

If $|S| = n$, then $|P(S)| = 2^n$

Another way of building a new set from two old ones is the Cartesian product of two sets.

Definition

Let S, T be sets. The *Cartesian product* $S \times T$ is defined as the set of tuples with elements from S, T, i.e

$$S \times T = \{(s, t) : s \in S \text{ and } t \in T\}.$$

This can also be extended inductively to a finite family of sets.

$S = T = \mathbb{R}$

$S \times T = \mathbb{R}^2 = \text{plane}.$

Ordered set

$\mathbb{R}$

Definition

A *relation* R on a set X is a subset of $X \times X$. A relation $\leq$ is called a *partial order* on X if it satisfies

1. reflexivity: $x \leq x$
2. transitivity: if $x \leq y$ and $y \leq z$, then $x \leq z$
3. anti-symmetry: if $x \leq y$ and $y \leq x$, then $x = y$.

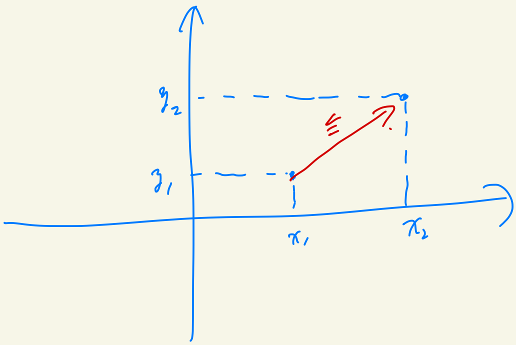
The pair $(X, \leq)$ is called a *partially ordered set*.

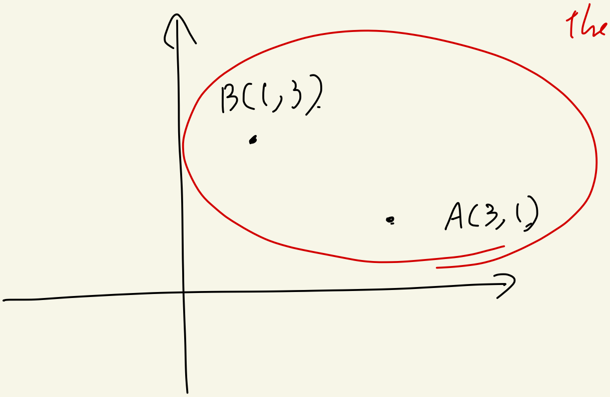
A *chain* or *totally ordered set* $C \subseteq X$ is a subset with the property $x \leq y$ or $y \leq x$ for any $x, y \in C$.

chain is a subset on which order is always determined between any two points.

e.g. $\mathbb{R}^2$ define relation on $\mathbb{R}^2$ by

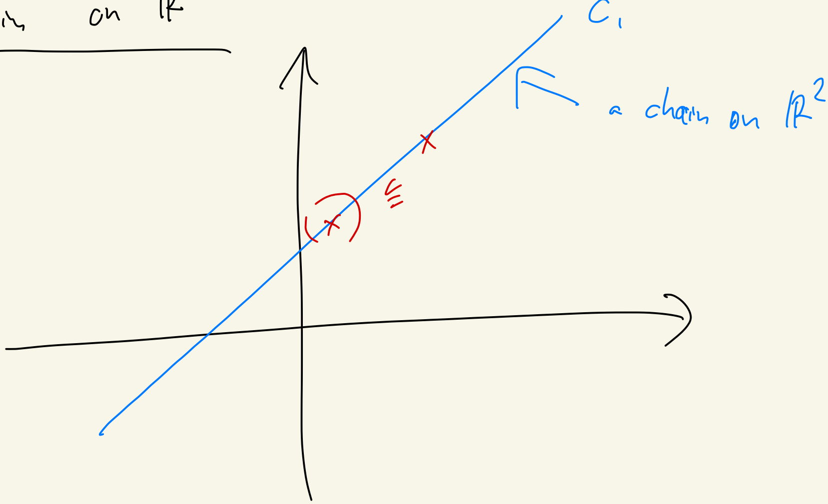
$(x_1, y_1) \leq (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$.





there is no order between these two points.

Chain on $\mathbb{R}^2$.



Example

The real numbers with the usual ordering, $(\mathbb{R}, \leq)$ are totally ordered.

Example

The power set of a set X with the ordering given by $\subseteq$, $(\mathcal{P}(X), \subseteq)$ is a partially ordered set.

$P(X)$ is not totally ordered

e.g. $X = \{a, b, c\}$

$\{a\}, \{b, c\}$

no order between them!

Example

Let $X = \{a, b, c, d\}$. What is $\mathcal{P}(X)$? Find a chain in $\mathcal{P}(X)$.

$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{c, d\}, \{b, d\}, \{a, c\}, \{a, d\}, \{a, b, c\},$
$\{b, c, d\}, \{a, b, d\}, \{a, c, d\}, X\}$

$$\underbrace{\emptyset \subset \{a\} \subset \{a, b\} \subset \{a, b, c\} \subset X}_{\text{chain}}$$

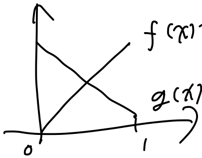
Example

Consider the set $C([0,1],\mathbb{R}) := \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$.

For two functions $f, g \in C([0,1],\mathbb{R})$, we define the ordering as $f \leq g$ if $f(x) \leq g(x)$ for $x \in [0,1]$. Then $(C([0,1],\mathbb{R}), \leq)$ is a partially ordered set.

not totally ordered

Can you think of a chain that is a subset of $(C([0,1],\mathbb{R})$?



$f(x) = x$

$g(x) = 1 - x$

no order

↳ example

1). constant functions.

2) $f_n(x) = f(x) + n$

Then $\{f_n(x)\}$ is a chain

3) $\{ f_a(x) = ax^2 : a \in \mathbb{R} \}$

Definition

A non-empty partially ordered set $(X, \leq)$ is *well-ordered* if every non-empty subset $A \subseteq X$ has a mimimum element.

Example:

$(\mathbb{N}, \leq)$ is... well-ordered.

$(\mathbb{R}, \leq)$ is... not well-ordered

$\left.\begin{array}{l} \mathbb{R} \text{ itself} \\ (a,b) \end{array}\right\}$ don't have minimum.

Definition

Let $(X, \leq)$ be a partially ordered set and $S \subseteq X$.

Then $x \in X$ is an *upper bound* for S if for all $s \in S$ we have $s \leq x$.
Similarly, $y \in X$ is a *lower bound* for S if for all $s \in S$, $y \leq s$.

If there exists an upper bound for S, we call S *bounded above* and if there exists a lower bound for S, we call S *bounded below*. If S is bounded above and bounded below, we say S is *bounded*.

We can also ask if there exists a least upper bound or a greatest lower bound.

Definition

Let $(X, \leq)$ be a partially ordered set and $S \subseteq X$.

We call $x \in X$ *least upper bound* or *supremum*, denoted $x = \sup S$, if x is an upper bound and for any other upper bound $y \in X$ of S we have $x \leq y$.

Likewise, $x \in X$ is the *greatest lower bound* or *infimum* for S, denoted $x = \inf S$, if it is a lower bound and for any other lower bound $y \in X$, $y \leq x$.

? $S = X = \mathbb{R}$

Note that the supremum and infimum of a bounded set do not necessarily need to exist. However, if they do exists they are unique, which justifies the article *the* (exercise). Nevertheless, the reals have a remarkable property, which we will take as an axiom.

$S = (\underset{\text{inf}}{0}, 1)$, $-S = (-1, \underset{\text{sup}}{0})$

Completeness Axiom

Let $S \subseteq \mathbb{R}$ be bounded above. Then there exists $r \in \mathbb{R}$ such that $r = \sup S$, i.e. S has a least upper bound.

i.e. the supremum exists

By setting $S' = -S := \{-s : s \in S\}$ and noting $\inf S = -\sup S'$, we obtain a similar statement for infima if S is bounded below. As mentioned above, this property is fairly special, for example it fails for the rationals.

Example

Let $S = \{q \in \mathbb{Q} : x^2 < 7\}$. Then S is bounded above in $\mathbb{Q}$, but there exists no least upper bound in $\mathbb{Q}$.

Solve $x^2 = 7$, $x = \pm\sqrt{7} \notin \mathbb{Q}$

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There is a nice alternative characterization for suprema in the real numbers.

Proposition

Let $S \subseteq \mathbb{R}$ be bounded above. Then $r = \sup S$ if and only if r is an upper bound and for all $\epsilon > 0$ there exists an $s \in S$ such that $r - \epsilon < s$.

Proof. $(\Rightarrow)$ if part

Suppose r is an upper bound and for $\forall \varepsilon > 0$, $\exists s \in S$ s.t. $r - \varepsilon < s$.

Suppose $r \neq \sup S$. Since r is an upper bound, $r > \sup S$.

Let $\varepsilon = r - \sup S > 0$. By assumption, $\exists s \in S$ s.t. $r - \varepsilon < s$.

Then $s > r - (r - \sup S) = \sup S$.

This is a contradiction.

Proof. ($\Leftarrow$) only if part. Suppose $r = \sup S$.

Suppose that $\exists\, \varepsilon > 0$, for $\forall s \in S$, $\underbrace{r - \varepsilon \geq s}$.

$r - \varepsilon$ is an upper bound of S.

However, $r - \varepsilon$ is smaller than $\sup S$. This is a contradiction.

Using the same trick, we may obtain a similar result for infima.

Example

Consider $S = \{1/n : \; n \in \mathbb{N}\}$. Then $\sup S = 1$ and $\inf S = 0$.

Functions

Definition

A function f from a set X to a set Y is a subset of $X \times Y$ with the properties:

1. For every $x \in X$, there exists a $y \in Y$ such that $(x, y) \in f$
2. If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

X is called the *domain* of f.

How does this connect to other descriptions of functions you may have seen?

Example

For a set X, the identity function is:

$$1_X : X \to X, \quad x \mapsto x$$

Definition (Image and pre-image)

Let $f\colon X \to Y$ and $A \subseteq X$ and $B \subseteq Y$.

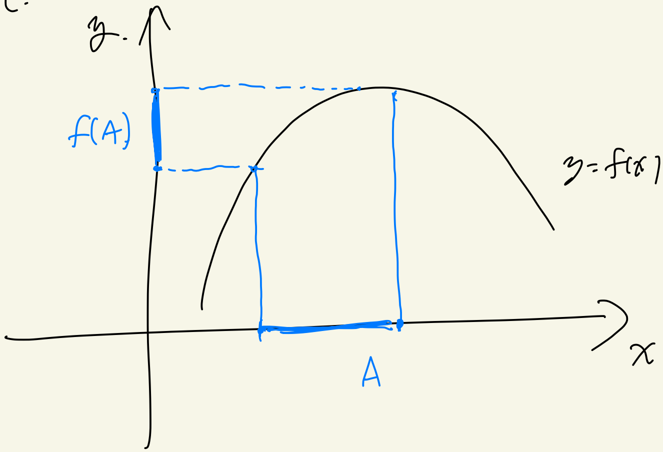
- The *image* of f is the set $f(A) := \{f(x) : x \in A\}$.
- The *pre-image* of f is the set $f^{-1}(B) := \{x : f(x) \in B\}$.

Helpful way to think about it for proofs:

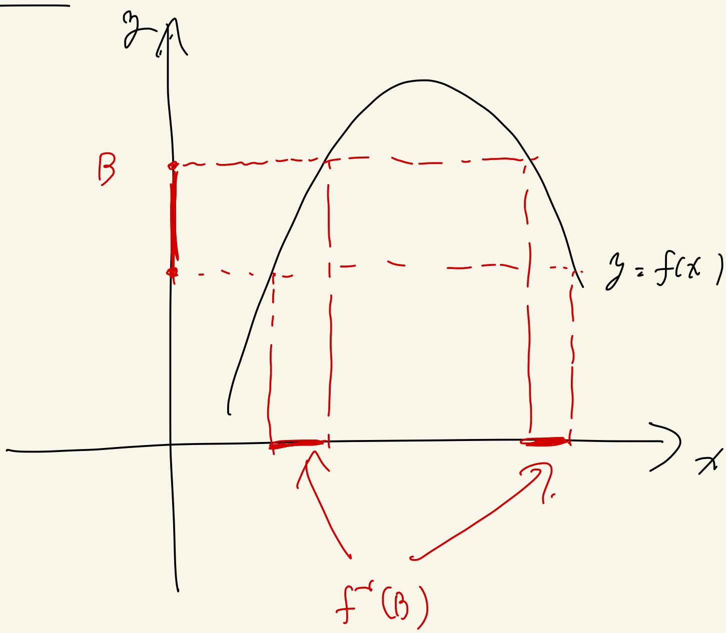
Image: If $y \in f(A)$, then $y \in Y$, and there exists an $x \in A$ such that $y = f(x)$.

Pre-image: If $x \in f^{-1}(B)$, then $x \in X$ and $f(x) \in B$.

Image.



Preimage.



Definition (Surjective, injective and bijective)

Let $f\colon X \to Y$, where X and Y are sets. Then

- f is *injective* if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$ $\Leftrightarrow$ if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
- f is *surjective* if for every $y \in Y$, there exists an $x \in X$ such that $y = f(x)$ $\Leftrightarrow f(X) = Y$
- f is *bijective* if it is both injective and surjective

Example

Let $f\colon X \to Y$, $x \mapsto x^2$.

f is surjective if $X = \mathbb{R}$, $Y = \mathbb{R}_{\geq 0}$

f is injective if $X = \mathbb{R}_{\geq 0}$

f is bijective if $X = Y = \mathbb{R}_{\geq 0}$

f is neither surjective nor injective if $X = \mathbb{R}$, $Y = \mathbb{R}$

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