

# Module 3: Set theory and metrics

## Operational math bootcamp



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# Outline

- More on set theory
- Cardinality of sets
- Metrics and norms

# Recall

## Definition (Image and pre-image)

Let  $f: X \rightarrow Y$  and  $A \subseteq X$  and  $B \subseteq Y$ .

- The *image* of  $f$  is the set  $f(A) := \{f(x) : x \in A\}$ .
- The *pre-image* of  $f$  is the set  $f^{-1}(B) := \{x : f(x) \in B\}$ .

## Definition (Surjective, injective and bijective)

Let  $f: X \rightarrow Y$ , where  $X$  and  $Y$  are sets. Then

- $f$  is *injective* if  $x_1 \neq x_2$  implies  $f(x_1) \neq f(x_2)$
- $f$  is *surjective* if for every  $y \in Y$ , there exists an  $x \in X$  such that  $y = f(x)$
- $f$  is *bijective* if it is both injective and surjective

## Proposition

Let  $f: X \rightarrow Y$  and  $A \subseteq X$ . Prove that  $A \subseteq f^{-1}(f(A))$ , with equality iff  $f$  is injective.

*Proof.*

# Cardinality

Intuitively, the *cardinality* of a set  $A$ , denoted  $|A|$ , is the number of elements in the set. For sets with only a finite number of elements, this intuition is correct. We call a set with finitely many elements finite.

We say that the empty set has cardinality 0 and is finite.

## Proposition

*If  $X$  is finite set of cardinality  $n$ , then the cardinality of  $\mathcal{P}(X)$  is  $2^n$ .*

*Proof.*



## Definition

Two sets  $A$  and  $B$  have same cardinality,  $|A| = |B|$ , if there exists bijection  $f: A \rightarrow B$ .

## Example

Which is bigger,  $\mathbb{N}$  or  $\mathbb{N}_0$ ?



# Cantor-Schröder-Bernstein

## Definition

We say that the cardinality of a set  $A$  is less than the cardinality of a set  $B$ , denoted  $|A| \leq |B|$  if there exists an injection  $f: A \rightarrow B$ .

## Theorem (Cantor-Bernstein)

Let  $A, B$ , be sets. If  $|A| \leq |B|$  and  $|B| \leq |A|$ , then  $|A| = |B|$ .

## Example

$$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$$

Proof that  $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$ :

## Definition

Let  $A$  be a set.

- ①  $A$  is *finite* if there exists an  $n \in \mathbb{N}$  and a bijection  $f: \{1, \dots, n\} \rightarrow A$
- ②  $A$  is *countably infinite* if there exists a bijection  $f: \mathbb{N} \rightarrow A$
- ③  $A$  is *countable* if it is finite or countably infinite
- ④  $A$  is *uncountable* otherwise

## Example

The rational numbers are countable, and in fact  $|\mathbb{Q}| = |\mathbb{N}|$ .

*Proof.* First we show  $|\mathbb{N}| \leq |\mathbb{Q}^+|$ .

Next, we show that  $|\mathbb{Q}^+| \leq |\mathbb{N} \times \mathbb{N}|$ .

We can extend this to  $\mathbb{Q}$  as follows:

## Theorem

The cardinality of  $\mathbb{N}$  is smaller than that of  $(0, 1)$ .

## Proof.

First, we show that there is an injective map from  $\mathbb{N}$  to  $(0, 1)$ .

Next, we show that there is no surjective map from  $\mathbb{N}$  to  $(0, 1)$ . We use the fact that every number  $r \in (0, 1)$  has a binary expansion of the form  $r = 0.\sigma_1\sigma_2\sigma_3\dots$  where  $\sigma_i \in \{0, 1\}$ ,  $i \in \mathbb{N}$ . □

## Proof.

Now we suppose in order to derive a contradiction that there does exist a surjective map  $f$  from  $\mathbb{N}$  to  $(0, 1)$ , i.e. for  $n \in \mathbb{N}$  we have  $f(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\dots$ . This means we can list out the binary expansions, for example like

$$f(1) = 0.00000000\dots$$

$$f(2) = 0.1111111111\dots$$

$$f(3) = 0.0101010101\dots$$

$$f(4) = 0.1010101010\dots$$

We will construct a number  $\tilde{r} \in (0, 1)$  that is not in the image of  $f$ . □



## Proof.

Define  $\tilde{r} = 0.\tilde{\sigma}_1\tilde{\sigma}_2\dots$ , where we define the  $n$ th entry of  $\tilde{r}$  to be the the opposite of the  $n$ th entry of the  $n$ th item in our list:

$$\tilde{\sigma}_n = \begin{cases} 1 & \text{if } \sigma_n(n) = 0, \\ 0 & \text{if } \sigma_n(n) = 1. \end{cases}$$

Then  $\tilde{r}$  differs from  $f(n)$  at least in the  $n$ th digit of its binary expansion for all  $n \in \mathbb{N}$ . Hence,  $\tilde{r} \notin f(\mathbb{N})$ , which is a contradiction to  $f$  being surjective. This technique is often referred to as Cantor's diagonal argument. □

## Proposition

$(0,1)$  and  $\mathbb{R}$  have the same cardinality.

## Proof.



We have shown that there are different sizes of infinity, as the cardinality of  $\mathbb{N}$  is infinite but still smaller than that of  $\mathbb{R}$  or  $(0,1)$ . In fact, we have

$$|\mathbb{N}| < |\mathbb{N}_0| < |\mathbb{Z}| < |\mathbb{Q}| < |\mathbb{R}|.$$

Because of this, there are special symbols for these two cardinalities: The cardinality of  $\mathbb{N}$  is denoted  $\aleph_0$ , while the cardinality of  $\mathbb{R}$  is denoted  $\mathfrak{c}$ .

In fact there are many other cardinalities, as the following theorem shows:

### Theorem (Cantor's theorem)

*For any set  $A$ ,  $|A| < |\mathcal{P}(A)|$ .*

# Metric Spaces

## Definition (Metric)

A *metric* on a set  $X$  is a function  $d : X \times X \rightarrow \mathbb{R}$  that satisfies:

- (a) Positive definiteness:
- (b) Symmetry:
- (c) Triangle inequality:

A set together with a metric is called a metric space.

## Example ( $\mathbb{R}^n$ with the Euclidean distance)

## Definition (Norm)

A *norm* on an  $\mathbb{F}$ -vector space  $E$  is a function  $\| \cdot \| : E \rightarrow \mathbb{R}$  that satisfies:

- (a) Positive definiteness:
- (b) Homogeneity:
- (c) Triangle inequality:

A vector space with a norm is called a *normed space*. A normed space is a metric space using the metric  $d(x, y) = \|x - y\|$ .

## Example ( $p$ -norm on $\mathbb{R}^n$ )

The  $p$ -norm is defined for  $p \geq 1$  for a vector  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  as

The infinity norm is the limit of the  $p$ -norm as  $p \rightarrow \infty$ , defined as



## Example ( $p$ -norm on $C([0, 1]; \mathbb{R})$ )

If we look at the space of continuous functions  $C([0, 1]; \mathbb{R})$ , the  $p$ -norm is

and the  $\infty$ -norm (or sup norm) is

## Definition

A subset  $A$  of a metric space  $(X, d)$  is *bounded* if there exists  $M > 0$  such that  $d(x, y) < M$  for all  $x, y \in A$ .

## Definition

Let  $(X, d)$  be a metric space. We define the *open ball* centred at a point  $x_0 \in X$  of radius  $r > 0$  as

$$B_r(x_0) := \{x \in X : d(x, x_0) < r\}.$$

## Example

In  $\mathbb{R}$  with the usual norm (absolute value), open balls are symmetric open intervals, i.e.

## Example: Open ball in $\mathbb{R}^2$ with different metrics

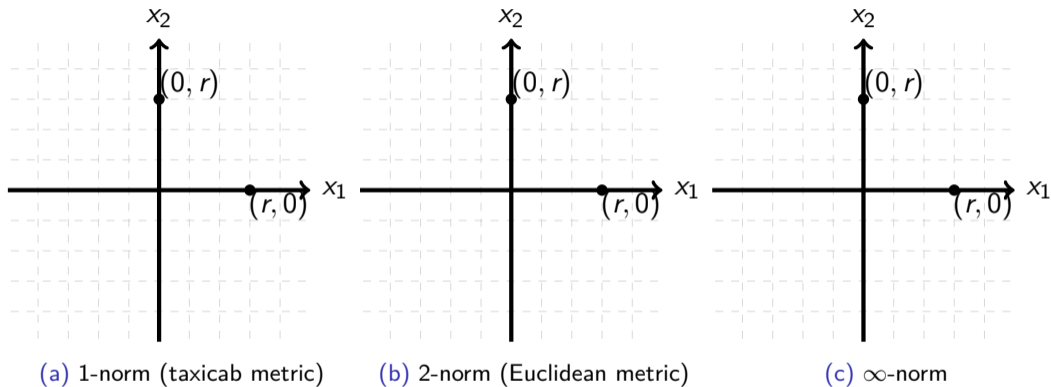


Figure:  $B_r(0)$  for different metrics

# References

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