# Module 4: Metric Spaces II Operational math bootcamp



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# Outline

- Open and closed sets
- Sequences
  - Cauchy sequences
  - subsequences



#### Definition (Open and closed sets)

Let (X, d) be a metric space.

- A set  $U \subseteq X$  is open if for every  $x \in U$  there exists  $\epsilon > 0$  such that  $B_{\epsilon}(x) \subseteq U$ .
- A set  $F \subseteq X$  is *closed* if  $F^c := X \setminus F$  is open.

#### Note:

#### Proposition

Let (X, d) be a metric space.

- 1 Let  $A_1, A_2 \subseteq X$ . If  $A_1$  and  $A_2$  are open, then  $A_1 \cap A_2$  is open.
- **2** If  $A_i \subseteq X$ ,  $i \in I$  are open, then  $\bigcup_{i \in I} A_i$  is open.



*Proof.* (1) Let  $A_1, A_2 \subseteq X$ . If  $A_1$  and  $A_2$  are open, then  $A_1 \cap A_2$  is open.

(2) If  $A_i \subseteq X$ ,  $i \in I$  are open, then  $\bigcup_{i \in I} A_i$  is open.



Using DeMorgan, we immediately have the following corollary:

#### Corollary

Let (X, d) be a metric space.

- **1** Let  $A_1, A_2 \subseteq X$ . If  $A_1$  and  $A_2$  are closed, then  $A_1 \cup A_2$  is closed.
- **2** If  $A_i \subseteq X$ ,  $i \in I$  are closed, then  $\bigcap_{i \in I} A_i$  is closed.



#### Definition (Interior and closure)

Let  $A \subseteq X$  where (X, d) is a metric space.

- The *closure* of A is  $\overline{A} :=$
- The *interior* of A is  $\overset{\circ}{A} :=$
- The *boundary* of A is  $\partial A :=$

#### Example

Let  $X = (a, b] \subseteq \mathbb{R}$  with the ordinary (Euclidean) metric. Then



Let 
$$A \subseteq X$$
 where  $(X, d)$  is a metric space. Then  $\mathring{A} = A \setminus \partial A$ .



Let (X, d) be a metric space and  $A \subseteq X$ .  $\overline{A}$  is closed and  $\overset{\circ}{A}$  is open.

Proof.

## Remark

In fact, 
$$\mathring{A} = \bigcup \{ U : U \text{ is open and } U \subseteq A \}$$
 and  $\overline{A} = \bigcap \{ F : F \text{ is closed and } A \subseteq F \}$ .





## Definition (Sequence)

Let (X, d) be a metric space. A sequence is an ordered list of points  $x_n$ ,  $n \in \mathbb{N}$ , in X, denoted  $(x_n)_{n \in \mathbb{N}}$ . We say that a sequence  $(x_n)_{n \in \mathbb{N}}$  converges to a point  $x \in X$  if



## **Recall:** $\overline{A} =$

#### Proposition

Let (X, d) be a metric space, and let  $A \subseteq X$ . Then  $\overline{A}$  is equal to the set of points in X which are limits of a sequence in A.



#### Corollary

A set  $F \subseteq X$ , where (X, d) is a metric space, is closed if and only if every sequence in F which converges in X converges to a point in F.

#### **Remark:**



# Cluster points of a set

#### Definition

Let (X, d) be a metric space and  $A \subseteq X$ . A point  $x \in X$  is a *cluster point* of A (also called accumulation point) if for every  $\epsilon > 0$ ,  $B_{\epsilon}(x)$  contains infinitely many points in A.



 $x \in X$  is a cluster point of  $A \subseteq X$  where (X, d) is a metric space if and only if there exists a sequence of points  $x_n \in A$ ,  $n \in \mathbb{N}$ , such that  $x_n \to x$ .



Combining the previous result with the limit characterization of closure gives the following:

# Corollary For $A \subseteq X$ , (X, d) a metric space, we have $\overline{A} = A \cup \{x \in X : x \text{ is a cluster point of } A\}.$



# **Cauchy sequences**

## Definition (Cauchy sequence)

Let (X, d) be a metric space. A sequence denoted  $(x_n)_{n \in \mathbb{N}} \in X$  is called a *Cauchy* sequence if



Let (X, d) be a metric space, and let  $(x_n)_{n \in \mathbb{N}}$  be a convergent sequence in X. Then  $(x_n)_{n \in \mathbb{N}}$  is Cauchy.



#### Definition

A metric space where every Cauchy sequence converges (to a point in the space) is called *complete*.

#### Example:

#### Proposition

Let (X, d) be a metric space, and let  $Y \subseteq X$ .

(i) If X is complete and if Y is closed in X, then Y is complete.

(ii) If Y is complete, then it is closed in X.

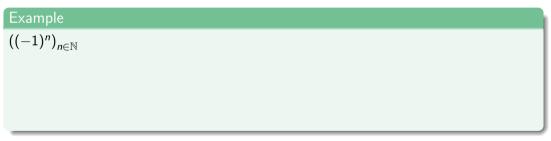




# **Subsequences**

#### Definition

Let  $(x_n)_{n\in\mathbb{N}}$  be a sequence in a metric space (X, d). Let  $(n_k)_{k\in\mathbb{N}}$  be a sequence of natural numbers with  $n_1 < n_2 < \cdots$ . The sequence  $(x_{n_k})_{k\in\mathbb{N}}$  is called a *subsequence* of  $(x_n)_{n\in\mathbb{N}}$ . If  $(x_{n_k})_{k\in\mathbb{N}}$  converges to  $x \in X$ , we call x a *subsequential limit*.



A sequence  $(x_n)_{n \in \mathbb{N}}$  in a metric space (X, d) converges to  $x \in X$  if and only if every subsequence of  $(x_n)_{n \in \mathbb{N}}$  also converges to x.



#### Proof continued



## References

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