Module 4: Metric Spaces II

Operational math bootcamp

Statistical Sciences
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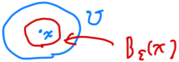
Outline

- $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$.
- $d(x,y) = d(y,x)$
- $d(x,y) + d(y,z) \geq d(x,z)$

- Open and closed sets
- Sequences
  - Cauchy sequences
  - subsequences



Definition (Open and closed sets)

Let (X, d) be a metric space.

- A set $U \subseteq X$ is *open* if for every $x \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$.
- A set $F \subseteq X$ is *closed* if $F^c := X \setminus F$ is open.

Note: $\emptyset$, X are both open and closed

Proposition

Let (X, d) be a metric space.

1. *Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are open, then $A_1 \cap A_2$ is open.* → can be extended to finite intersection
2. *If $A_i \subseteq X$, $i \in I$ are open, then $\cup_{i \in I} A_i$ is open.*

can be infinite

Proof. (1) Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are open, then $A_1 \cap A_2$ is open.

Let $x \in A_1 \cap A_2$.

Since A_1 is open, $\exists\, \varepsilon_1 > 0$ s.t. $B_{\varepsilon_1}(x) \subset A_1$

Since A_2 is open, $\exists\, \varepsilon_2 > 0$ s.t. $B_{\varepsilon_2}(x) \subset A_2$.

Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Then $B_\varepsilon(x) \subset B_{\varepsilon_1}(x) \subset A_1$

and $B_\varepsilon(x) \subset B_{\varepsilon_2}(x) \subset A_2$. Thus $B_\varepsilon(x) \subset A_1 \cap A_2$.

(2) If $A_i \subseteq X$, $i \in I$ are open, then $\cup_{i \in I} A_i$ is open.

Let $x \in \bigcup_{i \in I} A_i$. $\exists\, i$ s.t. $x \in A_i$.

Since A_i is open $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset A_i \subset \bigcup_{i \in I} A_i$

Using DeMorgan, we immediately have the following corollary:

Corollary

Let (X, d) be a metric space.

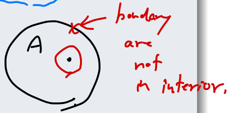
1. *Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are closed, then $A_1 \cup A_2$ is closed.*
2. *If $A_i \subseteq X$, $i \in I$ are closed, then $\cap_{i \in I} A_i$ is closed.*

Definition (Interior and closure)

Let $A \subseteq X$ where (X, d) is a metric space.

- The *closure* of A is $\overline{A} := \{x \in X : \forall \varepsilon > 0,\ B_\varepsilon(x) \cap A \neq \emptyset\}$
- The *interior* of A is $\mathring{A} := \{x \in A : \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subset A\}$.
- The *boundary* of A is $\partial A :=$

$$\{x \in X : \forall \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \cap A \neq \emptyset \text{ and } B_\varepsilon(x) \cap A^c \neq \emptyset\}.$$



Example

Let $X = (a, b] \subseteq \mathbb{R}$ with the ordinary (Euclidean) metric. Then

$$\overline{X} = [a,b], \quad \mathring{X} = (a,b), \quad \partial X = \{a, b\}$$

check!

Proposition

Let $A \subseteq X$ where (X, d) is a metric space. Then $\mathring{A} = A \setminus \partial A$.

Proof. We show $\mathring{A} \subset A \setminus \partial A$ and $A \setminus \partial A \subset \mathring{A}$ separately.

(Proof of $\mathring{A} \subset A \setminus \partial A$)

Let $x \in \mathring{A}$. By definition, $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset A$. contradiction.

Suppose $x \in \partial A$. By definition, $B_\varepsilon(x) \cap A^c \neq \phi$

This is a contradiction, Therefore $x \in A \setminus \partial A$.

(Proof of $A \setminus \partial A \subset \mathring{A}$)

Let $x \in A \setminus \partial A$. Since $x \notin \partial A$, $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \cap A = \phi$ or $B_\varepsilon(x) \cap A^c = \phi$.

Since $x \in A$, $B_\varepsilon(x) \cap A \neq \phi$. Thus, $B_\varepsilon(x) \cap A^c = \phi \Longleftrightarrow B_\varepsilon(x) \subset A$.

Therefore, $x \in \mathring{A}$ //.

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smallest closed set $\supset A$

Proposition

Let (X, d) be a metric space and $A \subseteq X$. $\overline{A}$ is closed and $\mathring{A}$ is open. largest open set $\subset A$

Proof. $\mathring{A}$ is the largest open set $\subset A$ is obvious from definition,

To show $\bar{A}$ is closed, we need to show $\bar{A}^c$ is open.

Let $x \in \bar{A}^c$. Then by definition of $\bar{A}$, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \cap A = \emptyset$.

We need to show $B_\varepsilon(x) \subset \bar{A}^c$. Let $y \in B_\varepsilon(x)$.

Then, let $\tilde{\varepsilon} = \varepsilon - d(x, y)$.

Then $B_{\tilde{\varepsilon}}(y) \subset B_\varepsilon(x) \subset A^c$. This means $y \in \bar{A}^c$.

Thus $B_\varepsilon(x) \subset \bar{A}^c$. //

Remark

In fact, $\mathring{A} = \bigcup\{U : U \text{ is open and } U \subseteq A\}$ and $\overline{A} = \bigcap\{F : F \text{ is closed and } A \subseteq F\}$.

Furthermore, $\bar{A}$ is the smallest closed set $\supset A$.

(Pf.) Let F be a closed set $F \supset A$.

We must show $F \supset \bar{A} \Leftrightarrow F^c \subset \bar{A}^c$.

Let $x \in F^c \subset A^c$. Since F^c is open, $\exists \varepsilon > 0$ s.t.

$B_\varepsilon(x) \subset F^c \subset A^c$.

Suppose $x \in \bar{A}$. Then by definition, $B_\varepsilon(x) \cap A \neq \phi$

This is a contradiction.

Therefore $x \notin \bar{A} \Leftrightarrow x \in \bar{A}^c$.

Sequences

Definition (Sequence)

Let (X, d) be a metric space. A *sequence* is an ordered list of points x_n, $n \in \mathbb{N}$, in X, denoted $(x_n)_{n\in\mathbb{N}}$. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ *converges* to a point $x \in X$ if

$$\forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \quad d(x_n, x) < \varepsilon \quad \forall n \geq n_\varepsilon$$

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Recall: $\overline{A} = \{ x \in X : \forall \varepsilon > 0, \ B_\varepsilon(x) \cap A \neq \emptyset \}$.

Proposition

Let (X, d) be a metric space, and let $A \subseteq X$. Then $\overline{A}$ is equal to the set of points in X which are limits of a sequence in A.

Proof. $\overline{A} = \{ x \in X : \exists \{x_n\} \subset A \text{ s.t. } x_n \to x \}$.

(Proof of $\subset$)

Let $x \in \overline{A}$. By definition, $\forall \varepsilon > 0$, $B_\varepsilon(x) \cap A \neq \emptyset$.

Let $\varepsilon = \frac{1}{n}$. Then $B_{1/n}(x) \cap A \neq \emptyset$.

Pick $x_n \in B_{1/n}(x) \cap A$. Then $x_n \in A$ and $d(x_n, x) < \frac{1}{n}$.

For any $\varepsilon > 0$, by taking $\frac{1}{n_\varepsilon} < \varepsilon \iff n_\varepsilon > \varepsilon^{-1}$. Then for any $n \geq n_\varepsilon$,

$$d(x_n, x) < \frac{1}{n} \leq \frac{1}{n_\varepsilon} < \varepsilon.$$

(Proof of $\supset$)

Let x be the limit of $\{x_n\} \subset A$.

Let $\varepsilon > 0$. By definition of convergence $\exists n_\varepsilon$ s.t.

$n \geq n_\varepsilon$ implies $d(x_n, x) < \varepsilon \iff x_n \in B_\varepsilon(x)$.

At the same time $x_n \in A$ by definition. Thus $x_n \in B_\varepsilon(x) \cap A$.

Therefore $B_\varepsilon(x) \cap A \neq \phi$. //

$\bar{A} = \{ x \in X : \exists \{x_n\} \subset A \text{ s.t. } x_n \to x \}$

$\bar{A}$ is the smallest closed $\supset A$

Corollary

A set $F \subseteq X$, where (X, d) is a metric space, is closed if and only if every sequence in F which converges in X converges to a point in F.

Remark: F is closed $\iff F = \bar{F}$

$\iff F = \{ x \in X : \exists \{x_n\} \in F \text{ s.t. } x_n \to x \}$.

Cluster points of a set

also in A but not a cluster point

Definition

Let (X, d) be a metric space and $A \subseteq X$. A point $x \in X$ is a *cluster point* of A (also called accumulation point) if for every $\epsilon > 0$, $B_\epsilon(x)$ contains infinitely many points in A.

$\iff$ for $\forall\, \varepsilon > 0$, $B_\varepsilon(x) \cap (A \setminus \{x\}) \neq \emptyset$

Proposition

$x \in X$ is a cluster point of $A \subseteq X$ where (X, d) is a metric space if and only if there exists a sequence of points $x_n \in A$, $x_n \neq x$, $n \in \mathbb{N}$, such that $x_n \to x$.

Proof.

x is cluster point $\iff$ $\exists x_n \in A$, $x_n \neq x$ s.t. $x_n \to x$.

$(\Rightarrow)$ By definition $\exists x_n \in B_{1/n}(x) \cap A$ s.t. $x_n \neq x$. since $|B_{1/n}(x) \cap A| = \infty$.

then $x_n \to x$ holds.

$(\Leftarrow)$ Suppose $\exists \{x_n\} \subset A$ s.t. $x_n \to x$, $x_n \neq x$.

For any $\varepsilon > 0$, $\exists n_\varepsilon$ s.t. $n \geq n_\varepsilon$ implies $d(x_n, x) < \varepsilon$.

Thus for any $n \geq n_\varepsilon$, $x_n \in B_\varepsilon(x) \cap A$.

therefore $|B_\varepsilon(x) \cap A| = \infty$.

Combining the previous result with the limit characterization of closure gives the following:

Corollary

For $A \subseteq X$, (X, d) a metric space, we have

$$\overline{A} = A \cup \{x \in X : x \text{ is a cluster point of } A\}.$$

why not $\overline{A} = \{\text{cluster points of } A\}$

also in A but not a cluster point

Cauchy sequences

Definition (Cauchy sequence)

Let (X, d) be a metric space. A sequence denoted $(x_n)_{n\in\mathbb{N}} \in X$ is called a *Cauchy sequence* if

$$\forall \varepsilon > 0, \ \exists n_\varepsilon \ \text{s.t.}, \quad n, m \geq n_\varepsilon \Rightarrow d(x_n, x_m) < \varepsilon.$$

Proposition

Let (X, d) be a metric space, and let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in X. Then $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

Proof. Let $x_n \to x$.

For $\forall \varepsilon > 0$, $\exists n_\varepsilon$ s.t. $d(x_n, x) < \frac{\varepsilon}{2}$.

By triangle inequality, for $\forall m, n \geq n_\varepsilon$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Definition

A metric space where every Cauchy sequence converges (to a point in the space) is called *complete*.

Example: $\mathbb{R}$, $\mathbb{R}^n$ with usual metric are complete.

Proposition

Let (X, d) be a metric space, and let $Y \subseteq X$.

(i) If X is complete and if Y is closed in X, then Y is complete.

(ii) If Y is complete, then it is closed in X.

Proof.

(i) Let $\{x_n\} \subset Y$ is Cauchy.

$\{x_n\} \subset X$ and X is complete.

So $x_n \to x \in X$.

Since Y is closed the limit of convergent sequence must be in Y.

(ii) Since Y is complete, every convergent sequence in Y converges to a point in Y.

This equivalent to say Y is closed.

Subsequences

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X,d). Let $(n_k)_{k\in\mathbb{N}}$ be a sequence of natural numbers with $n_1 < n_2 < \cdots$. The sequence $(x_{n_k})_{k\in\mathbb{N}}$ is called a *subsequence* of $(x_n)_{n\in\mathbb{N}}$. If $(x_{n_k})_{k\in\mathbb{N}}$ converges to $x \in X$, we call x a *subsequential limit*.

Example

$((-1)^n)_{n\in\mathbb{N}}$ $\quad n = 2m \quad (-1)^n = 1 \to 1$

Proposition

A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, d) converges to $x \in X$ if and only if every subsequence of $(x_n)_{n\in\mathbb{N}}$ also converges to x.

Proof. only if part is trivial

(if part) Suppose x_n does not converge to x.

$\exists\, \varepsilon > 0$ s.t. $\exists\, n_k$ $d(x_{n_k}, x) \geq \varepsilon$. contradiction

By assumption $x_{n_k} \to x$.

there exists k_ε s.t. $k \geq k_\varepsilon \Rightarrow d(x_{n_k}, x) < \varepsilon$.

this is a contradiction.

Proof continued

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