Exercises for Module 5: Metric Spaces III

1. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. Prove that

$$f \text{ is Lipschitz continuous } \Rightarrow f \text{ is uniformly continuous } \Rightarrow f \text{ is continuous.}$$

Provide examples to show that the other directions do not hold.

(1) f is Lipschitz $\Rightarrow$ f is uniformly continuous

Suppose $f: X \to Y$ is Lipschitz with Lipschitz constant $K > 0$.
Let $\varepsilon > 0$ arbitrary. Choose $\delta = \varepsilon/K > 0$. Then if $x_1, x_2 \in X$ s.t. $d_X(x_1, x_2) < \delta = \varepsilon/K$, then $d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) < K\, \varepsilon/K = \varepsilon$. Thus f is uniformly cont. by def.

↳ by def of Lipschitz cont.

(2) Example of f that is unif cont but not Lipschitz.

Let $f(x) = \sqrt{x}$, $f: [0,1] \to [0,1]$

For $\varepsilon > 0$, choose $\delta = \varepsilon$. Then for any $x, y \in [0,1]$, if $|x-y| < \delta = \varepsilon^2$, then

$|\sqrt{x} - \sqrt{y}|^2 \leq |\sqrt{x} - \sqrt{y}|\,|\sqrt{x} + \sqrt{y}| = |x-y| < \varepsilon^2 \implies |\sqrt{x} - \sqrt{y}| < \varepsilon$

$\therefore f(x) = \sqrt{x}$ is unif. cont. on $[0,1]$

However, f is not Lipschitz.

Proof: Suppose in order to derive a contradiction that it is.

Then $\forall x, y \in [0,1]$, $|\sqrt{x} - \sqrt{y}| \leq K|x-y|$. Take $y = 0 \Rightarrow \sqrt{x} \leq \frac{K}{x} \Rightarrow \frac{1}{\sqrt{x}} \leq K$.

But $\lim_{x \to 0} \frac{1}{\sqrt{x}} = \infty \not\leq K$, which is a contradiction. $\therefore f$ is not Lipschitz

(3) f is unif cont. $\Rightarrow$ f is cont

This is clear from the definitions (using the ε-δ def of continuity).

Take δ to be the one from the def of uniformly continuous, and we are done.

(4) Example of a function which is continuous but not uniformly cont.

Choose $f: [0,\infty) \to [0,\infty)$, $f(x) = x^2$.

We know that f is continuous (prove it using ε-δ is you like).

Suppose in order to derive a contradiction that it is uniformly continuous.

Then for any $\varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall x, y \in X$ with $|x-y| < \delta$, $|x^2 - y^2| < \varepsilon$.

$\Rightarrow |x-y|\,|x+y| < \varepsilon$. Choose $\varepsilon = 1$ and $y = x + \frac{\delta}{2}$ (okay since $|x-y| = \frac{\delta}{2} < \delta$)

Then $\frac{\delta}{2}\left|x + x + \frac{\delta}{2}\right| < 1$

$\Rightarrow \frac{\delta}{2}\left(2x + \frac{\delta}{2}\right) < 1$

$\Rightarrow x\delta + \frac{\delta^2}{4} < 1$ $\Rightarrow$ Choose $x = 1/\delta$

$\Rightarrow 1 + \frac{\delta^2}{4} < 1$

Contradiction.

$\therefore f(x) = x^2$ not unif. cont.

2. Show that the function $f(x) = \frac{1}{2}\left(x + \frac{5}{x}\right)$ has a unique fixed point on $(0, \infty)$. What is it? (Hint: you will have to restrict the interval.)

We need $|f(x) - f(y)| \le k|x-y|$ for $k \in [0,1)$ & $x, y \in X$. We can pick $X \subset (0,\infty)$.

$$|f(x)-f(y)| = \left|\tfrac{1}{2}\left(x+\tfrac{5}{x}\right) - \tfrac{1}{2}\left(y+\tfrac{5}{y}\right)\right|$$
$$= \tfrac{1}{2}\left|x-y+\tfrac{5}{x}-\tfrac{5}{y}\right|$$
$$= \tfrac{1}{2}\left|x-y+\tfrac{5y-5x}{xy}\right|$$
$$= \tfrac{1}{2}\left|x-y-5\tfrac{x-y}{xy}\right|$$
$$= \tfrac{1}{2}|x-y|\left|1-\tfrac{5}{xy}\right|$$

So we need $\frac{1}{2}\left|1-\frac{5}{xy}\right| \le k$, $k \in [0,1)$. Take $k = 4/5$.

$$\Rightarrow \left|1-\tfrac{5}{xy}\right| \le 8/5$$
$$\Rightarrow -\tfrac{8}{5} \le 1-\tfrac{5}{xy} \le \tfrac{8}{5}$$

$-\frac{8}{5} \le 1-\frac{5}{xy}$ $\Rightarrow \frac{-13}{5} \le \frac{-5}{xy}$ $\Rightarrow \frac{13}{5} \ge \frac{5}{xy}$ $\Rightarrow xy \ge 25/13$

$1-\frac{5}{xy} \le \frac{8}{5}$ $\Rightarrow \frac{-5}{xy} \le \frac{4}{5}$ $\Rightarrow \frac{-25}{4} \le xy$ always true

If $x=y$, need $x^2 \ge \frac{25}{13}$

If $x=y$, need $x \ge \frac{5}{\sqrt{13}}$.

Proof: Let $X = \left[\frac{5}{\sqrt{13}}, \infty\right)$. X is complete since it is a closed subset of $\mathbb{R}$. Let $x, y \in X$. Then

$$|f(x)-f(y)| \overset{?}{=} \left|\tfrac{1}{2}\left(x-\tfrac{5}{x}\right)-\tfrac{1}{2}\left(y-\tfrac{5}{y}\right)\right| = \tfrac{1}{2}\left|x-\tfrac{5}{x}-y+\tfrac{5}{y}\right|$$
$$= \tfrac{1}{2}|x-y|\left|1-\tfrac{5}{xy}\right|$$
$$\le \tfrac{1}{2}|x-y|\left|1-\tfrac{5}{5/\sqrt{13}\cdot 5/\sqrt{13}}\right|$$
$$= \tfrac{1}{2}|x-y|\left|1-\tfrac{13}{5}\right|$$
$$= \tfrac{1}{2}\tfrac{8}{5}|x-y|$$
$$= 4/5\,|x-y|$$

Thus f is a contraction w/ constant $k = 4/5$.

$\therefore$ By the contraction mapping Thm, f has a unique fixed point in $\left[\frac{5}{\sqrt{13}}, \infty\right)$. (It is $\sqrt{5}$)

To justify that there is no other fixed point on $\left(0, \frac{5}{\sqrt{13}}\right)$, we note that

$$f\left(\tfrac{5}{\sqrt{13}}\right) = \tfrac{1}{2}\left(\tfrac{5}{\sqrt{13}}+\sqrt{13}\right) > \tfrac{5}{\sqrt{13}}$$

and since the function is decreasing on $(0, 5/\sqrt{13})$ $\left(f'(x) = \frac{1}{2}(1-5x^{-2}) < 0 \text{ if } x < 5/\sqrt{13}\right)$,

3. Prove the following: If two metrics are strongly equivalent then they are equivalent.

Proof. Let X be a set and d_1, d_2 be two metrics on X. Suppose they are strongly equivalent, i.e. for every $x, y \in X$ $\exists \alpha, \beta > 0$ s.t.

$$\alpha d_1(x,y) \le d_2(x,y) \le \beta d_1(x,y).$$

Let f be the identity map from (X, d_1) to (X, d_2). We show it is continuous using $\varepsilon - \delta$ definition.

Let $\varepsilon > 0$ be arbitrary. Choose $\delta = \varepsilon/\beta$. Then if $d_1(x,y) < \delta = \varepsilon/\beta$, we have $d_2(f(x), f(y)) = d_2(x,y) \overset{\text{by strong eq.}}{\le} \beta d_1(x,y) < \beta \,\varepsilon/\beta = \varepsilon$, so f is cont.

Similarly, for the id. map from (X, d_2) to (X, d_1): let $\varepsilon > 0$. Choose $\delta = \alpha\varepsilon$. Then $d_1(x,y) \underset{\text{strong eq.}}{\le} \frac{1}{\alpha} d_2(x,y) < \frac{1}{\alpha}\alpha\varepsilon = \varepsilon$, so it is continuous as well.

4. Let (X, d) be a metric space and $\{A_i\}_{i\in I}$ be a collection of subsets of X. Show that

$$\bigcup_{i\in I} \overline{A_i} \subseteq \overline{\bigcup_{i\in I} A_i}.$$

Show that if the collection is finite, the two sets are equal.

First we show $\bigcup_{i\in I} \overline{A_i} \subseteq \overline{\bigcup_{i\in I} A_i}$. Let $x \in \bigcup_{i\in I} \overline{A_i}$. Then $\exists \tilde{i} \in I$ s.t.

$$x \in \overline{A_{\tilde{i}}} \Rightarrow \forall \varepsilon > 0 \; B_\varepsilon(x) \cap A_{\tilde{i}} \neq \emptyset.$$

$$\Rightarrow \forall \varepsilon > 0 \; B_\varepsilon(x) \cap \Big(\bigcup_{i\in I} A_i\Big) \neq \emptyset$$

$$\Rightarrow x \in \Big(\overline{\bigcup_{i\in I} A_i}\Big)$$

Next, suppose the collection is finite. We show $\overline{\bigcup_{i=1}^n A_i} \subseteq \bigcup_{i=1}^n \overline{A_i}$.

First, we note that $\bigcup_{i=1}^n \overline{A_i}$ is closed. By the remark from class, $\overline{\bigcup_{i=1}^n A_i} = \bigcap \{F : F \text{ is closed and } \bigcup_{i=1}^n A_i \subseteq F\}$

Since $\bigcup_{i=1}^n A_i \subseteq \bigcup_{i=1}^n \overline{A_i}$, we conclude $\overline{\bigcup_{i=1}^n A_i} \subseteq \bigcup_{i=1}^n \overline{A_i}$.

5. Let (X, d) be a metric space and $\{A_i\}_{i\in I}$ be a collection of subsets of X. Prove that

$$\overline{\bigcap_{i\in I} A_i} \subseteq \bigcap_{i\in I} \overline{A_i}.$$

Find a counterexample that shows that equality is not necessarily the case.

Since $A_i \subseteq \overline{A_i} \Rightarrow \bigcap_{i\in I} A_i \subseteq \bigcap_{i\in I} \overline{A_i}$.

Since $\bigcap_{i\in I} \overline{A_i}$ is closed, $\overline{\bigcap_{i\in I} A_i} \subseteq \{F : F \text{ closed and } \bigcap_{i\in I} A_i \subseteq F\} \subseteq \bigcap_{i\in I} \overline{A_i}$.

Counterexample where $\bigcap_{i\in I} \overline{A_i} \not\subseteq \overline{\bigcap_{i\in I} A_i}$:

Let $A_1 = [0,1)$, $A_2 = (1,2]$, d = Euclidean metric on $\mathbb{R}$.

Then $\overline{A_1} = [0,1]$, $\overline{A_2} = [1,2]$, so $\bigcap_{i=1,2} \overline{A_i} = \{1\}$.

But $\overline{\left(\bigcap_{i=1,2} A_i\right)} = \overline{\emptyset} = \emptyset$.

6. Let (X, d) be a metric space and $A \subseteq X$ be dense. Show that if $A \subseteq B \subseteq X$, then B is dense as well.

Let $A \subseteq X$ be dense. Then $\overline{A} = X$. We want to show that $A \subseteq B \subseteq X \Rightarrow \overline{B} = X$.

Clearly $\overline{B} \subseteq X$, so we show $X \subseteq \overline{B}$.

Let $A \subseteq B$. Then $A \subseteq B \subseteq \overline{B}$.

Since $\overline{B}$ is a closed set that contains A, $\overline{A} \subseteq \overline{B}$ by Remark 3.16.

Thus $\overline{A} = X \subseteq \overline{B}$, so $\overline{B} = X$ and B is dense.