Exercises for Module 5: Metric Spaces III

1. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. Prove that

$$f \text{ is Lipschitz continuous } \Rightarrow f \text{ is uniformly continuous } \Rightarrow f \text{ is continuous.}$$

Provide examples to show that the other directions do not hold.

2. Show that the function $f(x) = \frac{1}{2}\left(x + \frac{5}{x}\right)$ has a unique fixed point on $(0, \infty)$. What is it? (Hint: you will have to restrict the interval.)

3. Prove the following: If two metrics are strongly equivalent then they are equivalent.

4. Let (X, d) be a metric space and $\{A_i\}_{i \in I}$ be a collection of subsets of X. Show that

$$\bigcup_{i \in I} \overline{A_i} \subseteq \overline{\bigcup_{i \in I} A_i}.$$

Show that if the collection is finite, the two sets are equal.

5. Let (X, d) be a metric space and $\{A_i\}_{i \in I}$ be a collection of subsets of X. Prove that

$$\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}.$$

Find a counterexample that shows that equality is not necessarily the case.

6. Let (X, d) be a metric space and $A \subseteq X$ be dense. Show that if $A \subseteq B \subseteq X$, then B is dense as well.