Module 5: Metric spaces III

Operational math bootcamp

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Last time

Looked at open and closed sets.

Discussed sequences, which includes Cauchy sequences and subsequences:

- Convergent sequence: $x_n \to x \Leftrightarrow \forall \epsilon > 0 \; \exists \, n_\epsilon \in \mathbb{N}$ s.t. $d(x_n, x) < \epsilon$ for all $n \geq n_\epsilon$
- Cauchy sequence: $\forall \epsilon > 0 \; \exists \, n_\epsilon \in \mathbb{N}$ s.t. $d(x_n, x_m) < \epsilon$ for all $n, m \geq n_\epsilon$
- Proved that a convergent sequence is Cauchy
- Discussed complete metric spaces, where all Cauchy sequences converge (like $\mathbb{R}$ with the usual metric, absolute value)
- Proved that a sequence converges to x if and only if all subsequences converge to x

Outline for today

- Continuity
- Equivalent metrics
- Density and separability

Continuity

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$, and let $f : X \to Y$. f is *continuous* at x_0 if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X that converges to x_0, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

We say that f is continuous if it is continuous at every point in X.

Theorem

Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$, and let $f : X \to Y$. The following are equivalent:

(i) *f is continuous at x_0*

(ii) *for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0))) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$*

(iii) *for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$*

(i) f is continuous at x_0
(ii) for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0))) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$
(iii) for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$

Proof. (i) $\Rightarrow$ (ii)

(ii) $\Rightarrow$ (iii)

(iii) $\Rightarrow$ (i)

Corollary

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f\colon X \to Y$. The following are equivalent:

(i) f is continuous

(ii) if $U \subseteq Y$ is open, then $f^{-1}(U)$ is open

(iii) if $F \subseteq Y$ is closed, then $f^{-1}(F)$ is closed

We need the following results about sets and functions:
Let X and Y be sets and $f : X \to Y$. Let $A, B \subseteq Y$. Then

1. $A \subseteq B \implies f^{-1}(A) \subseteq f^{-1}(B)$
2. $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.
(i) $\Rightarrow$ (ii):

(ii) $\Rightarrow$ (i)

(ii) $\Rightarrow$ (iii)

(iii) $\Rightarrow$ (ii)

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.

- f is *uniformly continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that for every $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$, we have $d_Y(f(x_1), f(x_2))) < \epsilon$
- f is *Lipschitz continuous* if there exists a $K > 0$ such that for every $x_1, x_2 \in X$ we have $d_Y(f(x_1), f(x_2))) \leq K d_X(x_1, x_2)$

Proposition

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.

$$f \text{ is Lipschitz continuous} \Rightarrow \text{f is uniformly continuous} \Rightarrow \text{f is continuous}$$

Proof is one of your exercises.

Contraction Mapping Theorem

Definition

Let (X, d) be a metric space and let $f\colon X \to X$. We say that $x^* \in X$ is a *fixed point* of f if $f(x^*) = x^*$.

Definition

Let (X, d) be a metric space and let $f\colon X \to X$. f is a *contraction* if there exists a constant $k \in [0, 1)$ such that for all $x, y \in X$, $d(f(x), f(y))) \leq kd(x, y)$.

Observe that a function is a contraction if and only if it is Lipschitz continuous with constant $K < 1$.

Theorem (Contraction Mapping Theorem)

Suppose that $f\colon X \to X$ is a contraction and the metric space X is complete. Then f has a unique fixed point x^*.

Example

Let $f\colon \left[-\frac{1}{3}, \frac{1}{3}\right] \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ be defined by the mapping $x \mapsto x^2$. Assume we use the standard Euclidean metric, $d(x, y) = |x - y|$. f has a unique fixed point because

Equivalent metrics

Definition (Equivalent metrics)

Two metrics d_1 and d_2 on a set X are *equivalent* if the identity maps from (X, d_1) to (X, d_2) and from (X, d_2) to (X, d_1) are continuous.

Proposition

Two metrics d_1, d_2 on a set X are equivalent if and only if they have the same open sets or the same closed sets.

Definition

Two metrics d_1 and d_2 on a set X are *strongly equivalent* if for every $x, y \in X$, there exists constants $\alpha > 0$ and $\beta > 0$ such

$$\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y).$$

If two metrics are strongly equivalent then they are equivalent. The proof of this is one of the exercises.

Example

We show that the Euclidean distance (induced by 2-norm) and the metric induced by the ∞-norm are equivalent on $\mathbb{R}^n$.

Can you think of an example that we've seen of a metric that isn't equivalent to the Euclidean metric?

Density

Definition

Let (X, d) be a metric space. A subset $A \subseteq X$ is called *dense* if $\overline{A} = X$.

Why are dense sets important?

Examples

1. $(\mathbb{R}, |\cdot|)$

2. Let X be a set and define $d\colon X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

Definition

A metric space (X, d) is *separable* if it contains a countable dense subset.

Example:

Example

Define $\ell_\infty = \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{R}, \sup_{n\in\mathbb{N}} |x_n| < \infty\}$, the space of bounded real valued sequences. Endow ℓ_∞ with a metric induced by the supremum norm, namely $d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sup_{n\in\mathbb{N}} |x_n - y_n|$. Then ℓ_∞ is **not separable** with respect to the topology induced by this metric.

Note: this proof relies on content from the cardinality section. Specifically, Cantor's theorem gives us that $|A| < |\mathcal{P}(A)|$ for any set A.

Proof.

Proof continued.

References

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