# Module 5: Metric spaces III

## Operational math bootcamp

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# Last time

Looked at open and closed sets.

Discussed sequences, which includes Cauchy sequences and subsequences:

- Convergent sequence: $x_n \to x \Leftrightarrow \forall \epsilon > 0 \; \exists \, n_\epsilon \in \mathbb{N}$ s.t. $d(x_n, x) < \epsilon$ for all $n \geq n_\epsilon$
- Cauchy sequence: $\forall \epsilon > 0 \; \exists \, n_\epsilon \in \mathbb{N}$ s.t. $d(x_n, x_m) < \epsilon$ for all $n, m \geq n_\epsilon$
- Proved that a convergent sequence is Cauchy
- Discussed complete metric spaces, where all Cauchy sequences converge (like $\mathbb{R}$ with the usual metric, absolute value)
- Proved that a sequence converges to $x$ if and only if all subsequences converge to $x$

# Outline for today

- Continuity
- Equivalent metrics
- Density and separability

# Continuity

## Definition

Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, let $x_0 \in X$, and let $f: X \to Y$. $f$ is *continuous* at $x_0$ if for every sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ that converges to $x_0$, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

We say that $f$ is continuous if it is continuous at every point in $X$.

## Theorem

*Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, let $x_0 \in X$, and let $f : X \to Y$. The following are equivalent:*

(i) *$f$ is continuous at $x_0$* (previous definition)

(ii) *for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$*

depends on $x_0$

(iii) *for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$*

(i) $f$ is continuous at $x_0$ If $x_n \to x_0$, then $f(x_n) \to f(x_0)$

(ii) for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0))) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$

(iii) for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$

*Proof.* (i) $\Rightarrow$ (ii) We prove the contrapositive.

$\exists\ \varepsilon > 0$, $\forall\ \delta > 0$, $\exists\ x_\delta \in X$ s.t. $d_X(x_\delta, x_0) < \delta$ and $d_Y(f(x_\delta), f(x_0)) \geq \varepsilon$.

Let $\delta = \frac{1}{n}$, we pick $x_n$ s.t. $d_X(x_n, x_0) < \frac{1}{n}$ and $d_Y(f(x_n), f(x_0)) \geq \varepsilon$.

Then, since $d_X(x_n, x_0) < \frac{1}{n}$, we have $x_n \to x_0$.

However $d_Y(f(x_n), f(x_0)) \geq \varepsilon$ for $\forall\ n$ implies $f(x_n) \not\to f(x_0)$.

Therefore, the contrapositive is proved.

(ii) $\Rightarrow$ (iii) Exercise. (Hint (iii) is just (ii) rephrased in a concise way.)

(iii) $\Rightarrow$ (i) Let $x_n \to x_0$.

Let $\varepsilon > 0$. Then by (iii), $\exists\, \delta > 0$ s.t. $B_\delta(x_0) \subset f^{-1}(B_\varepsilon(f(x_0)))$.

Since $x_n \to x_0$, $\exists\, n_\varepsilon$ s.t. $n \geq n_\varepsilon \Rightarrow d_X(x_n, x_0) < \delta$.

Note that $d_X(x_n, x_0) < \delta \Leftrightarrow x_n \in B_\delta(x_0) \subset f^{-1}(B_\varepsilon(f(x_0)))$

The last expression means $f(x_n) \in B_\varepsilon(f(x_0))$

$\Leftrightarrow d_Y(f(x_n), f(x_0)) < \varepsilon$.

Thus, we have shown, $\exists\, n_\varepsilon$ s.t. $n \geq n_\varepsilon \Rightarrow d_Y(f(x_n), f(x_0)) < \varepsilon$ //

## Corollary

Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and let $f : X \to Y$. The following are equivalent:

(i) $f$ is continuous on $X$

(ii) if $U \subseteq Y$ is open, then $f^{-1}(U)$ is open ← this is used as definition of continuity for general topological spaces.

(iii) if $F \subseteq Y$ is closed, then $f^{-1}(F)$ is closed

We need the following results about sets and functions:
Let $X$ and $Y$ be sets and $f\colon X \to Y$. Let $A, B \subseteq Y$. Then

1. $A \subseteq B \implies f^{-1}(A) \subseteq f^{-1}(B)$
2. $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$

(exercise)

*Proof.* Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and let $f\colon X \to Y$.
(i) $\Rightarrow$ (ii):

Let $V \subset Y$ be open. We want to show $f^{-1}(V)$ is open.

Let $x \in f^{-1}(V)$. Then $f(x) \in V$, which is open.

There exists $\varepsilon > 0$ s.t. $B_\varepsilon(f(x)) \subset V$

Since $f$ is continuous at $x$, $\exists\, \delta > 0$ s.t. $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x)))$

Since $B_\varepsilon(f(x)) \subset V$, we have $f^{-1}(B_\varepsilon(f(x))) \subset f^{-1}(V)$.

Thus, we have shown $B_\delta(x) \subset f^{-1}(V)$. Thus $f^{-1}(V)$ is open. //

(ii) $\Rightarrow$ (i) We will prove the equivalent definition (iii) of continuity at $x_0$.

Let $x_0 \in X$ and $\varepsilon > 0$.

Since $B_\varepsilon(f(x_0))$ is open, $f^{-1}(B_\varepsilon(f(x_0)))$ is open (we used (ii)).

Then, $\exists\, \delta > 0$, s.t. $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x_0)))$.

This is exactly the equivalent definition (iii).

(ii) $\Rightarrow$ (iii)

Let $F \subset Y$ be closed.

Then $Y \setminus F$ is open.

By (ii) $f^{-1}(Y \setminus F)$ is open.

However, $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$.

Therefore, $f^{-1}(F)$ is closed $/\!/$

(iii) $\Rightarrow$ (ii)

Exercise.

## Definition

Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and let $f: X \to Y$. does not depend on $x_1, x_2$.

- $f$ is *uniformly continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that for every $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$, we have $d_Y(f(x_1), f(x_2))) < \epsilon$
- $f$ is *Lipschitz continuous* if there exists a $K > 0$ such that for every $x_1, x_2 \in X$ we have $d_Y(f(x_1), f(x_2))) \leq K d_X(x_1, x_2)$

## Proposition

Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and let $f: X \to Y$.

$$f \text{ is Lipschitz continuous } \Rightarrow \text{ f is uniformly continuous } \Rightarrow \text{ f is continuous}$$

Proof is one of your exercises.

example.)

$f(x) = \frac{1}{x}$ , $x > 0$

$f(x)$ is not uniformly continuous.

Let $\varepsilon > 0$.

$$\left| \frac{1}{x} - \frac{1}{x_0} \right| = \frac{1}{x x_0} |x - x_0| < \varepsilon.$$

$\to \infty$ as $x_0 \to 0$

$$\Longleftrightarrow \quad |x - x_0| < \varepsilon x x_0 \to 0$$

as $x_0 \to 0$

$\parallel$

$\delta$?

# Contraction Mapping Theorem

**Definition**

Let $(X, d)$ be a metric space and let $f\colon X \to X$. We say that $x^* \in X$ is a *fixed point* of $f$ if $f(x^*) = x^*$.

**Definition**

Let $(X, d)$ be a metric space and let $f\colon X \to X$. $f$ is a *contraction* if there exists a constant $k \in [0, 1)$ such that for all $x, y \in X$, $d(f(x), f(y))) \leq kd(x, y)$.

Observe that a function is a contraction if and only if it is Lipschitz continuous with constant $K < 1$.

**Theorem (Contraction Mapping Theorem)**

*Suppose that $f\colon X \to X$ is a contraction and the metric space $X$ is complete. Then $f$ has a unique fixed point $x^*$.*

## Example

Let $f\colon \left[-\frac{1}{3}, \frac{1}{3}\right] \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ be defined by the mapping $x \mapsto x^2$. Assume we use the standard Euclidean metric, $d(x, y) = |x - y|$. $f$ has a unique fixed point because

(i) $\left[-\frac{1}{3}, \frac{1}{3}\right]$ is complete

(ii) $$|x^2 - y^2| = \underbrace{|x+y|}_{\leq \frac{2}{3}}\,|x-y| \leq \underbrace{\frac{2}{3}\,|x-y|}_{\text{contraction!}}$$

Thus $f$ is a contraction.

Therefore, the existence of a fixed point is guaranteed.

# Equivalent metrics

**Definition (Equivalent metrics)**

Two metrics $d_1$ and $d_2$ on a set $X$ are *equivalent* if the identity maps from $(X, d_1)$ to $(X, d_2)$ and from $(X, d_2)$ to $(X, d_1)$ are continuous.

**Proposition**

Two metrics $d_1$, $d_2$ on a set $X$ are equivalent if and only if they have the same open sets or the same closed sets.

## Definition

Two metrics $d_1$ and $d_2$ on a set $X$ are *strongly equivalent* if for every $x, y \in X$, there exists constants $\alpha > 0$ and $\beta > 0$ such

$$\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y).$$

$d_1$ and $d_2$ are essentially same upto constant factor

If two metrics are strongly equivalent then they are equivalent. The proof of this is one of the exercises.

## Example

We show that the Euclidean distance (induced by 2-norm) and the metric induced by the $\infty$-norm are equivalent on $\mathbb{R}^n$.

$$\|x-z\|_2 = \sqrt{\sum_{i=1}^{n}(x_i-z_i)^2}, \quad \|x-z\|_\infty = \max_i |x_i-z_i|$$

$$\|x-z\|_2 = \sqrt{\sum_{i=1}^{n}(x_i-z_i)^2} \leq \sqrt{n \cdot \max_i |x_i-z_i|^2} = \sqrt{n} \cdot \|x-z\|_\infty$$

$$\|x-z\|_\infty = \max_i |x_i-z_i| = \sqrt{\max_i |x_i-z_i|^2}$$

$$\leq \sqrt{\sum_{i=1}^{n} |x_i-z_i|^2} = \|x-z\|_2.$$

Can you think of an example that we've seen of a metric that isn't equivalent to the Euclidean metric?

exercise.

# Density

**Definition**

Let $(X, d)$ be a metric space. A subset $A \subseteq X$ is called *dense* if $\overline{A} = X$.

Definition of $\overline{A} = \{x \in X : \forall \varepsilon > 0,\ B_\varepsilon(x) \cap A \neq \emptyset\}$.

From this we can say $A \subset X$ is dense

$\iff \forall x \in X, \forall \varepsilon > 0, B_\varepsilon(x) \cap A \neq \emptyset.$

Why are dense sets important?

## Examples

1. $(\mathbb{R}, |\cdot|)$

   $\mathbb{Q}$ is dense in $\mathbb{R}$.

2. Let $X$ be a set and define $d \colon X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

the only dense set in $X$ is $X$ itself

(exercise)

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### Definition

A metric space $(X, d)$ is *separable* if it contains a countable dense subset.

**Example:** $\mathbb{R}$ is separable since $\mathbb{Q}$ is countable and dense.

## Example

Define $\ell_\infty = \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{R},\ \sup_{n\in\mathbb{N}} |x_n| < \infty\}$, the space of bounded real valued sequences. Endow $\ell_\infty$ with a metric induced by the supremum norm, namely $d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sup_{n\in\mathbb{N}} |x_n - y_n|$. Then $\ell_\infty$ is **not separable** with respect to the topology induced by this metric.

Note: this proof relies on content from the cardinality section. Specifically, Cantor's theorem gives us that $|A| < |\mathcal{P}(A)|$ for any set $A$.

*Proof.* Suppose $S \subset \ell_\infty$ is a countable dense subset.

Let $S = \{ (x^k_n)_{n\in\mathbb{N}} : k \in \mathbb{N} \}$.

$k=1$ : $x^1_1, x^1_2, x^1_3, \ldots$

$k=2$ : $x^2_1, x^2_2, x^2_3, \ldots$

$k=3$ : $x^3_1, x^3_2, x^3_3, \ldots$

*Proof continued.*

Define a new sequence $(y_n)_{n\in\mathbb{N}}$ by

$$y_n = \begin{cases} 0 & \text{if } |x_n^n| > 1 \\ x_n^n + 1 & \text{if } |x_n^n| \le 1. \end{cases}$$

By definition, $|y_n| \le 2$ thus $(y_n)_{n\in\mathbb{N}} \in \ell_\infty$

Furthermore, definition of $y_n$ implies

$$d\left((y_n)_{n\in\mathbb{N}}, (x_n^k)_{n\in\mathbb{N}}\right) \geq 1 \text{ for } {}^\forall k.$$

By taking $0 < \varepsilon < 1$, then $B_\varepsilon((y_n)_{n\in\mathbb{N}}) \cap S = \phi$.

This contradicts to the assumption that $S$ is dense.

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(Proof ②)

For any $I \subset \mathbb{N}$, define a sequence $e^I = (e_n^I)_{n\in\mathbb{N}}$ by

$$e_n^I = \begin{cases} 1 & \text{if } n \in I \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $I \neq J$, ${}^\exists n$ s.t. $|e_n^I - e_n^J| = 1$

Therefore, $d(e^I, e^J) = 1$ if $I \neq J$.

Thus, if $\varepsilon \in (0, 1/2)$, $B_\varepsilon(e^I) \cap B_\varepsilon(e^J) = \phi$ if $I \neq J$.

So all $B_\varepsilon(e^I)$ are disjoint.

Let $S \subset l_\infty$ be a dense subset.

Then $B_\varepsilon(e^I) \cap S \neq \phi$ by definition.

But since $B_\varepsilon(e^I)$ are disjoint, these intersections are disjoint.

Thus

$$|S| \geq |\{I : I \subset \mathbb{N}\}| = |\mathcal{P}(\mathbb{N})| > |\mathbb{N}|.$$

Thus $S$ is uncountable.

# References

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