Module 6: Metric Spaces IV

Operational math bootcamp

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Outline

- Compactness
- Extra properties of $\mathbb{R}$
  - Right- and left-continuity
  - Lim sup and lim inf

Last time

Definition

Let (X, d) be a metric space. A subset $A \subseteq X$ is called *dense* if $\overline{A} = X$.

Definition

A metric space (X, d) is *separable* if it contains a countable dense subset.

Example

$\mathbb{R}$ is separable because $\mathbb{Q}$ is dense in $\mathbb{R}$ and countable.

Example

Define $\ell_\infty = \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{R},\ \sup_{n\in\mathbb{N}} |x_n| < \infty\}$, the space of bounded real valued sequences. Endow ℓ_∞ with a metric induced by the supremum norm, namely $d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sup_{n\in\mathbb{N}} |x_n - y_n|$. Then ℓ_∞ is **not separable** with respect to the topology induced by this metric.

Note: this proof relies on content from the cardinality section. Specifically, Cantor's theorem gives us that $|A| < |\mathcal{P}(A)|$ for any set A.

Proof.

Proof continued.

Compactness

Definition

Let (X, d) be a metric space and $K \subseteq X$. can be $|I| = \infty$

open covering

A collection $\{U_i\}_{i\in I}$ of open sets is called *open cover* of K if $K \subseteq \cup_{i\in I} U_i$.

The set K is called *compact* if for all open covers $\{U_i\}_{i\in I}$ there exists a finite subcover, meaning there exists an $n \in \mathbb{N}$ and $\{U_1, \ldots, U_n\} \subseteq \{U_i\}_{i\in I}$ such that $K \subseteq \cup_{i=1}^{n} U_i$.

Example

Let $S \subseteq X$ where (X, d) is a metric space. If S is finite, then it is compact.

Proof. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover of $S = \{x_i, i = 1, \dots, n\}$.

So, $S = \bigcup_{i=1}^{n} \{x_i\} \subset \bigcup_{\lambda \in \Lambda} U_\lambda$.

For each i, $\exists\, \lambda_i \in \Lambda$ s.t. $x_i \in U_{\lambda_i}$

Then $S = \bigcup_{i=1}^{n} \{x_i\} \subset \underbrace{\bigcup_{i=1}^{n} U_{\lambda_i}}_{\text{finite sub cover of } S}$

but $[0,1]$ is compact (will be discussed later)

Example

$(0, 1)$ is not compact.

Proof. Let $U_n = (\frac{1}{n}, 1-\frac{1}{n})$. $U_1 \subset U_2 \subset U_3 \subset \cdots$

Then $(0,1) = \bigcup_{n \geq 1}^{\infty} U_n$. Thus $\{U_n\}_{n \in \mathbb{N}}$ is an open cover of $(0,1)$.

However, for any finite subset $I \subset \mathbb{N}$, let M_I be the maximum integer in I, then

$$\bigcup_{i \in I} U_i = U_{M_I} = \left(\frac{1}{M_I}, 1-\frac{1}{M_I}\right) \subsetneq (0,1)$$

So, there is no finite subcover from $\{U_n\}_{n \in \mathbb{N}}$.

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Proposition

Let (X, d) be a metric space and take a non-empty subset $K \subseteq X$. The following holds:

1. If X is compact and K is closed, then K is compact (i.e. closed subsets of compact sets are compact).
2. If K is compact, then K is closed. $\Rightarrow$ any open set which is not closed is not compact.

Proof. (1) If X is compact and $K \subseteq X$ is closed, then K is compact

Let $K \subset \bigcup_{\lambda \in \Lambda} U_\lambda$, where U_λ is an open set.

Adding $\underbrace{K^c}_{\text{open}}$ to $\{U_\lambda\}_{\lambda \in \Lambda}$, it becomes an open cover of X.

Since X is compact, there exists finite subcover $\{K^c\} \cup \{U_{\lambda_i}\}_{i=1,\dots,n}$.

Since $K^c \cap K = \phi$, it means $\{U_{\lambda_i}\}_{i=1,\dots,n}$ covers K.

Therefore, K is compact.

(2) $K \subseteq X$ compact $\Rightarrow$ K is closed.

Suppose. $x_n \in K \to x \notin K$.

Since $x \notin K$ and $\bigcap_{\varepsilon>0} \overline{B_\varepsilon(x)} = \{x\}$. $K \cap \left(\bigcap_{\varepsilon>0} \overline{B_\varepsilon(x)} \right) = \phi$.

Thus $K \subset \left(\bigcap_{\varepsilon>0} \overline{B_\varepsilon(x)} \right)^c = \bigcup_{\varepsilon>0} \underbrace{\overline{B_\varepsilon(x)}^c}_{\text{open set}}$

By compactness of K, we can pick finite ε's

$\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n > 0$ s.t.

$K \subset \bigcup_{i=1}^{n} \overline{B_{\varepsilon_i}(x)}^c = \overline{B_{\varepsilon_n}(x)}^c$.

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Thus $K \cap \overline{B_{\varepsilon_n}(x)} = \phi$.

This contradicts with the assumption $x_n \in K \to x$.

Arbitrary compact metric spaces have some nice properties in general as the next proposition shows.

Proposition

A compact metric space (X, d) is complete and separable.

Also, just as we had a sequential characterization of the closure of a set in metric spaces, we similarly have a sequential characterization of compactness.

Theorem

Let (X, d) be a metric space. Then $K \subseteq X$ is compact with respect to the metric induced by d if and only if every sequence in K admits a subsequence converging to some point in K.

Compactness on $\mathbb{R}^n$

Theorem (Heine-Borel Theorem)

Let $K \subseteq \mathbb{R}^n$. Then K is compact with respect to the topology induced by the Euclidean distance if and only if it is closed and bounded.

A corollary of the last two theorems is the Bolzano-Weierstrass theorem.

Corollary (Bolzano-Weierstrass)

Any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

(Proof) Let $\{x_n\}$ be a bounded sequence. $\exists M>0$. s.t. $|x_n| \leq M$. for $\forall n$.

Let $K = \{ |x| \leq M \}$. By H-B, K is cpt.

Furthermore, by sequential characterization of compactness, $\{x_n\}$ admits convergent subsequence in K.

Proposition

Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $K \subseteq X$ is compact and let $f\colon K \to Y$ be continuous. Then $f(K)$ is compact.

Note: this is a generalization of the Extreme Value Theorem to metric spaces.

Recall from the set theory section:

If $f : X \to Y$:

1. $A \subseteq B \subseteq Y \Rightarrow f^{-1}(A) \subseteq f^{-1}(B)$ and $A \subseteq B \subseteq X \Rightarrow f(A) \subseteq f(B)$
2. $f^{-1}(\cup_{i\in I} A_i) = \cup_{i\in I} f^{-1}(A_i)$, where $A_i \subseteq Y \forall i \in I$
3. $f(\cup_{i\in I} A_i) = \cup_{i\in I} f(A_i)$, where $A_i \subseteq X \forall i \in I$
4. $A \subseteq X \Rightarrow A \subseteq f^{-1}(f(A))$
5. $B \subseteq Y \Rightarrow f(f^{-1}(B)) \subseteq B$

Proof. Let $f(K) \subset \bigcup_{\lambda \in \Lambda} U_\lambda$ be open cover.

Then $K \subset \bigcup_{\lambda \in \Lambda} \underbrace{f^{-1}(U_\lambda)}$.

open since f is cont. and U_λ is open.

By compactness of K, there exists subcover $\{f^{-1}(U_{\lambda_i})\}_{i=1,\dots,n}$.

Thus, $K \subset \bigcup_{i=1}^{n} f^{-1}(U_{\lambda_i})$.

Therefore,

$$f(K) \subset \bigcup_{i=1}^{n} f(f^{-1}(U_{\lambda_i}))$$

$$\subset \bigcup_{i=1}^{n} U_{\lambda_i}.$$

Extra properties of $\mathbb{R}$

Right and left continuous

Recall: $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|x_0 - y| < \delta$ implies $|f(x_0) - f(y)| < \epsilon$.

Definition

Let $f\colon \mathbb{R} \to \mathbb{R}$.

- f is *left continuous* at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$, such $|f(x_0) - f(x)| < \epsilon$ whenever $x_0 - \delta < x < x_0$. left of x_0
- f is *right continuous* at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$, such $|f(x_0) - f(x)| < \epsilon$ whenever $x_0 < x < x_0 + \delta$. right of x_0

We say that f is left continuous if it is left continuous at all points in the domain, and similar for right continuous.

Proposition

A function $f \colon \mathbb{R} \to \mathbb{R}$ is continuous if and only if it is left and right continuous.

Proof.

exercise.

Bounded sequences and monotone convergence

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. We call $(x_n)_{n\in\mathbb{N}}$ *bounded* if there exists an $M > 0$ such that $|x_n| < M$ for all $n \in \mathbb{N}$.

Theorem (Monotone convergence theorem)

(i) *Suppose $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence, i.e. $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, and that it is bounded (above). Then the sequence converges. Furthermore, $\lim_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} x_n$, where $\sup_{n\in\mathbb{N}} x_n := \sup\{x_n : n \in \mathbb{N}\}$.*

(ii) *Suppose $(x_n)_{n\in\mathbb{N}}$ is a decreasing sequence, i.e. $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$, which is bounded (below). Then the sequence converges and $\lim_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} x_n := \inf\{x_n : n \in \mathbb{N}\}$.*

Convention: $\sup A = \infty$ if $A \subseteq \mathbb{R}$ is not bounded above and $\inf A = -\infty$ if A is not bounded below.

Lemma

If $A \subseteq B \subseteq \mathbb{R}$ is non-empty, then $\inf A \leq \sup A$, $\sup A \leq \sup B$, and $\inf A \geq \inf B$.

The proof of this follows from the definition of greatest lower and least upper bound.

Definition

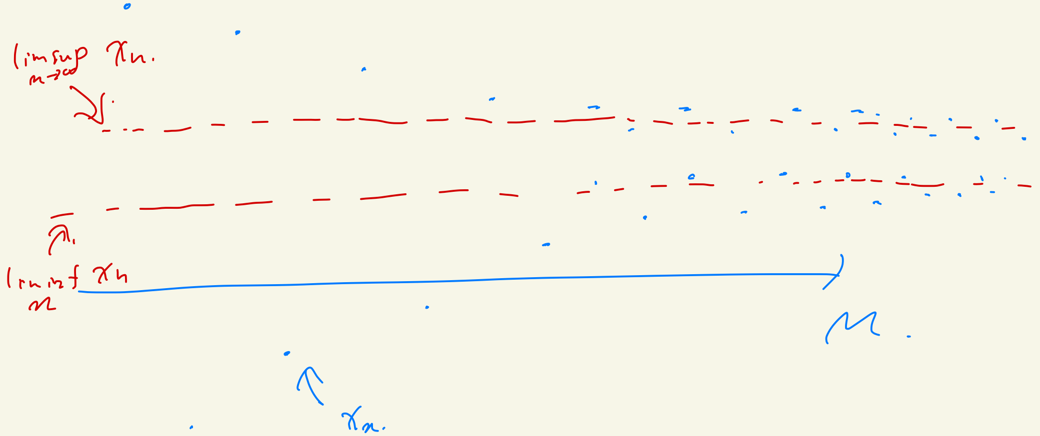
Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. We define the *limit superior* of $(x_n)_{n\in\mathbb{N}}$ as

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty}\left(\sup_{k\geq n} x_k.\right) = \inf_{n}\left(\sup_{k\geq n} x_k.\right)$$

Similarly we define the *limit inferior* of $(x_n)_{n\in\mathbb{N}}$ as

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty}\left(\inf_{k\geq n} x_k.\right) = \sup_{n.}\left(\inf_{k\geq n} x_k.\right)$$

If the sequence $(x_n)_{n\in\mathbb{N}}$ is not bounded above, then $\limsup_{n\to\infty} x_n = \infty$. Similarly, if the sequence $(x_n)_{n\in\mathbb{N}}$ is not bounded below, then $\liminf_{n\to\infty} x_n = -\infty$.


$\limsup_{n\to\infty} x_n$.
$\liminf_{n} x_n$
x_n.
M.

Proposition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$.

- The sequence of suprema, $s_n = \sup_{k\geq n} x_k$, is decreasing and the sequence of infima, $i_n = \inf_{k\geq n} x_k$, is increasing.
- The limit superior and the limit inferior of a bounded sequence always exist and are finite.

Proof.

exercise.

Theorem

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then the sequence converges to $x \in \mathbb{R}$ if and only if $\limsup_{n\to\infty} x_n = x = \liminf_{n\to\infty} x_n$.

Proof in notes.

We can extend this easily to a sequence of functions $f_n : X \to \mathbb{R}$ as follows:

Define $f = \limsup_{n\to\infty} f_n$ to be the function defined pointwise by $f(x) = \limsup_{n\to\infty}(f_n(x))$ and similar for the limit inferior.

There also exists a set theoretic version in terms of unions and intersections which you will encounter in probability.

References

Runde, Volker (2005). *A Taste of Topology*. Universitext. url: https://link.springer.com/book/10.1007/0-387-28387-0