Module 9: Linear Algebra III

Operational math bootcamp

Statistical Sciences
UNIVERSITY OF TORONTO

Ichiro Hashimoto

University of Toronto

July 24, 2024

Outline

- Adjoints, unitaries and orthogonal matrices
- Orthogonal decomposition
- Spectral theory
  - Eigenvalues and eigenvectors
  - Algebraic and geometric multiplicity of eigenvalues
  - Matrix diagonalization

Recall

Definition

Let V be an $\mathbb{F}$-vector space. A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ is called *inner product* on V if the following holds:

1. (Conjugate) symmetry: $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ for all $\mathbf{x}, \mathbf{y} \in V$, where $\overline{a}$ denotes the complex conjugate for $a \in \mathbb{C}$
2. Linearity in the first argument: $\langle \alpha \mathbf{x} + \beta \mathbf{y}, \mathbf{z} \rangle = \alpha \langle \mathbf{x}, \mathbf{z} \rangle + \beta \langle \mathbf{y}, \mathbf{z} \rangle$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha, \beta \in \mathbb{F}$
3. Positive definiteness: $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$

A vector space equipped with an inner product is called an *inner product space*.

Recall

Example

- Standard inner product on $\mathbb{R}^n$: $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$
- Standard inner product on $\mathbb{C}^n$: $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i \bar{y}_i$ for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$
- On the space of polynomials $\mathbb{P}_n(\mathbb{R})$: $\langle \boldsymbol{p}, \boldsymbol{q} \rangle = \int_{-1}^{1} p(x)q(x)\mathrm{d}x$ for $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}_n(\mathbb{R})$

Proposition

Let V be an inner product space. Then

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}\sqrt{\langle \mathbf{y}, \mathbf{y} \rangle}$$

for all $\mathbf{x}, \mathbf{y} \in V$.

Proposition

Let V be an inner product space. Then $\langle \cdot, \cdot \rangle$ induces a norm on V via $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ for all $\mathbf{x} \in V$.

Proof.

Note: With this identification the Cauchy-Schwarz inequality can be restated as:
$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in V$.

Example

The norm introduced by the standard inner product on $\mathbb{R}^n$ is the Euclidean distance.

Adjoint

Definition

Let U, V be inner product spaces and $S\colon U \to V$ be a linear map. The *adjoint* S^* of S is the linear map $S^*: V \to U$ defined such that

$$\langle S\mathbf{u}, \mathbf{v}\rangle_V = \langle \mathbf{u}, S^*\mathbf{v}\rangle_U \qquad \text{for all } \mathbf{u} \in U, \mathbf{v} \in V.$$

Proposition

Let U, V be inner product spaces and $S\colon U \to V$ be a linear map. Then S^* is unique and linear.

Proof. Let R, T both satisfies the condition of adjoint S^*.

$$\langle Su, v \rangle = \langle u, Rv \rangle = \langle u, Tv \rangle.$$

$$\langle u, (R-T)v \rangle = 0$$

$$\langle (R-T)v, u \rangle = 0$$

$$\therefore \langle (R-T)v, u \rangle = 0$$

Since this holds for any $u \in U$, we must have

$(R-T)v = 0$ for $\forall v \in V$.

$\therefore \quad Rv = Tv \text{ for } \forall v \in V$

Example

Define $S\colon \mathbb{R}^3 \to \mathbb{R}^2$ by $S\mathbf{x} = (2x_1 + x_3, -x_2)$. What is the adjoint operator S^*?

Use $\langle Sx, y \rangle = \langle x, S^* y \rangle$.

$$\text{LHS} = \langle (2x_1 + x_3, -x_2), (y_1, y_2) \rangle$$

$$= (2x_1 + x_3) y_1 - x_2 y_2$$

$$= x_1 \cdot (2y_1) + x_2 (-y_2) + x_3 \cdot y_1$$

$$= \langle (x_1, x_2, x_3), \underbrace{(2y_1, -y_2, y_1)}_{= S^* y} \rangle.$$

Thus

$$S^* y = \begin{pmatrix} 2y_1 \\ -y_2 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\therefore S^* = \begin{pmatrix} 2 & 0 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Proposition

Let $A \in M_{m\times n}(\mathbb{F})$ be a matrix and $T_A \colon \mathbb{F}^n \to F^m \colon \mathbf{x} \mapsto A\mathbf{x}$. Then, $T_A^*(\mathbf{x}) = A^*\mathbf{x}$, where $A^* \in M_{n\times m}(\mathbb{F})$ with $(A^*)_{ij} = \overline{A_{ji}}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

In particular, if $\mathbb{F} = \mathbb{R}$, the adjoint of the matrix is given by its transpose,denoted A^T, and if $\mathbb{F} = \mathbb{C}$, it is given by its conjugate transpose, denoted A^*.

$A^* = \bar{A}^T$

Proof for $\mathbb{R}$:

$$\langle Ax, y\rangle = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} A_{ij}\, x_j\right) y_i$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} A_{ij}\, x_j\, y_i$$

$$= \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{n} A_{ij}\, y_i\right)$$

$$= \langle x, A^T y\rangle$$

Definition

A matrix $O \in M_n(\mathbb{R})$ is called *orthogonal* if its inverse is given by its transpose, i.e. $O^T O = O O^T = I$.

A matrix $U \in M_n(\mathbb{C})$ is called *unitary* if the inverse is given by the conjugate transpose, i.e. $U^* U = U U^* = I$.

Let U be unitary.

$$\langle Ux, Uy \rangle = \langle x, \underbrace{U^* U}_{= Id} y \rangle = \langle x, y \rangle$$

U does not change inner product

Example

- Let $\varphi \in [0, 2\pi]$. Then
$$\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \quad \text{Rotation}$$
is an orthogonal matrix. What does it describe geometrically?
- The following is a unitary matrix:
$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = A$$

$$A^* = \bar{A}^T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}^T$$

$$A A^* = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{Id.} \qquad = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Statistical Sciences
UNIVERSITY OF TORONTO

Definition

Let $A \in M_n(\mathbb{F})$. We call *A self-adjoint* if $A^* = A$. In the case $\mathbb{F} = \mathbb{R}$, such an A is called *symmetric* and if $\mathbb{F} = \mathbb{C}$, such an A is called *Hermitian*.

Orthogonality and Gram-Schmidt

Definition

Two vectors $\mathbf{x}, \mathbf{y} \in V$ are called *orthogonal* if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, denoted $\mathbf{x} \perp \mathbf{y}$. We call them *orthonormal* if additionally the vectors are normalized, i.e. $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$. A basis $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of V is called *orthonormal basis (ONB)*, if the vectors are pairwise orthogonal and normalized.

Proposition

Let $\mathbf{x}_1, \ldots, \mathbf{x}_k \in V$ be orthonormal. Then the system of vectors is linearly independent.

Proof. Let $\sum_{i=1}^{k} \alpha_i x_i = 0$. Need to show $\alpha_1 = \cdots = \alpha_k = 0$.

Since $\sum_{i=1}^{k} \alpha_i x_i = 0$, $\left\| \sum_{i=1}^{k} \alpha_i x_i \right\|^2 = 0$.

Expanding the RHS,

$$0 = \sum_{i=1}^{k} |\alpha_i|^2 \|x_i\|^2 + \sum_{i \neq j} \alpha_i \bar{\alpha}_j \underbrace{\langle x_i, x_j \rangle}_{=0 \text{ since } x_i \perp x_j}$$

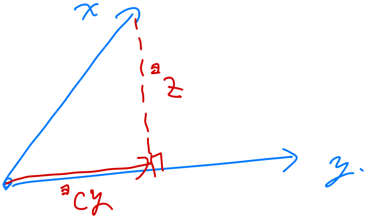
$$= \sum_{i=1}^{k} |\alpha_i|^2 \underbrace{\|x_i\|^2}_{=1} = \sum_{i=1}^{k} |\alpha_i|^2.$$

Therefore $\alpha_i = 0$ for $\forall i$.

Proposition (Orthogonal Decomposition)

Let $\mathbf{x}, \mathbf{y} \in V$ with $\mathbf{y} \neq 0$. Then, there exist $c \in F$ and $\mathbf{z} \in V$ such that $\mathbf{x} = c\mathbf{y} + \mathbf{z}$ with $\mathbf{y} \perp \mathbf{z}$.

Given a basis, we can obtain an ONB from it using the Gram-Schmidt algorithm by repeating this orthogonal decomposition.



Proposition (Gram-Schmidt Algorithm)

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in V$ be a system of linearly independent vectors. Define $\mathbf{y}_1 = \mathbf{x}_1 / \|\mathbf{x}_1\|$. For $i = 2, \ldots, n$ define $\mathbf{y}_j$ inductively by

$$\mathbf{y}_i = \frac{\mathbf{x}_i - \sum_{k=1}^{i-1} \langle \mathbf{x}_i, \mathbf{y}_k \rangle \mathbf{y}_k}{\|\mathbf{x}_i - \sum_{k=1}^{i-1} \langle \mathbf{x}_i, \mathbf{y}_k \rangle \mathbf{y}_k\|}.$$

Then the $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are orthonormal and

$$\operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_n\} = \operatorname{span}\{\mathbf{y}_1, \ldots, \mathbf{y}_n\}.$$

The proof is omitted but can be found in the book.

Recall: connection between matrices and linear maps

Multiplication by a matrix defines a linear map

Let $A \in M_{m \times n}$ be a fixed matrix. Then, we can define a linear map $T_A \colon \mathbb{F}^n \to \mathbb{F}^m$ via $T_A(\mathbf{v}) = A\mathbf{v}$, where we recall matrix vector multiplication $(A\mathbf{v})_i = \sum_{k=1}^{n} A_{ik} v_k$ for $i = 1, \ldots, m$.

Given a bases for U and V, $T : U \to V$ can be written as a matrix

Let $T \in \mathcal{L}(U, V)$ where U and V are vector spaces. Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ and $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be bases for U and V respectively. The matrix of T with respect to these bases is the $m \times n$ matrix $\mathcal{M}(T)$ with entries A_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$ defined by

$$T\mathbf{u}_k = A_{1k}\mathbf{v}_1 + \cdots + A_{mk}\mathbf{v}_m.$$

Eigenvalues

Definition

Given an operator $A\colon V \to V$ and $\lambda \in \mathbb{F}$, λ is called an *eigenvalue* of A if there exists a non-zero vector $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ such that

$$A\mathbf{v} = \lambda \mathbf{v}.$$

We call such $\mathbf{v}$ an *eigenvector* of A with eigenvalue λ. We call the set of all eigenvalues of A *spectrum* of A and denote it by $\sigma(A)$.

Motivation in terms of linear maps: Let $T\colon V \to V$ be a linear map, where V is a vector space. We would like to describe the action of this linear map in a particularly "nice" way: such that T acts only by scaling, i.e. $T\mathbf{v}_i = \lambda_i \mathbf{v}_i$ where $\lambda_i \in \mathbb{F}$ for $i = 1, \ldots, n$.

Finding eigenvalues

Note: here we will assume $\mathbb{F} = \mathbb{C}$, so that we are working on an algebraically closed field.

- Rewrite $A\mathbf{v} = \lambda\mathbf{v}$ as $(A-\lambda I)v = 0 \iff v \in \ker(A-\lambda I)$
  eigenspace.
- Thus, if λ is an eigenvalue, we can find the corresponding eigenvectors by finding the null space of $A - \lambda I$.
- The subspace null$(A - \lambda I)$ is called the *eigenspace*
- To find the eigenvalues of A, one must find the scalars λ such that null$(A - \lambda I)$ contains non-trivial vectors (i.e. not $\mathbf{0}$)
  If $\dim U = \dim V = n$, T is invertible
- Recall: We saw that $T \in \mathcal{L}(U, V)$ is injective if and only if null $T = \{\mathbf{0}\}$. $\iff$ T is injective.
- Thus λ is an eigenvalue if and only if $A - \lambda I$ is not invertible. Since $A-\lambda I$ is injective.
- Recall: $|A| \neq 0$ if and only if A is invertible.
- Thus λ is an eigenvalue if and only if

$\Downarrow$

$\det(A-\lambda I) = 0$

Theorem

The following are equivalent

1. $\lambda \in \mathbb{F}$ *is an eigenvalue of* A,
2. $(A - \lambda I)\mathbf{v} = 0$ *has a non-trivial solution,*
3. $|A - \lambda I| = 0$.

solving this polynomial equation to obtain eigenvalues of A.

Statistical Sciences
UNIVERSITY OF TORONTO

Characteristic polynomial

Definition

If A is an $n \times n$ matrix, $p_A(\lambda) = |A - \lambda I|$ is a polynomial of degree n called the *characteristic polynomial* of A.

To find the eigenvectors of A, one needs to find the roots of the characteristic polynomial.

Example

Find the eigenvalues of

$$A = \begin{bmatrix} 4 & -2 \\ 5 & -3 \end{bmatrix}.$$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & -2 \\ 5 & -3-\lambda \end{vmatrix} = (4-\lambda)(-3-\lambda) + 10$$

$$= \lambda^2 - \lambda - 12 + 10 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$$

$$\therefore \lambda = -1, 2.$$

Multiplicity

$p(\lambda) = \lambda^{2} (\lambda-1)^{3} (\lambda-2)^{1}$

Algebraic multiplicity of $\lambda = 0$ is 2

" " $\lambda = 1$ is 3

" " $\lambda = 2$ is 1

Definition

The multiplicity of the root λ in the characteristic polynomial is called the *algebraic multiplicity* of the eigenvalue λ. The dimension of the eigenspace null$(A - \lambda I)$ is called the *geometric multiplicity* of the eigenvalue λ.

||

# of vectors consist

a basis of $\ker(A - \lambda I)$.

Definition (Similar matrices)

Square matrices A and B are called *similar* if there exists an invertible matrix S such that

$$A = SBS^{-1}.$$

Similar matrices have the same characteristic polynomials and hence the same eigenvalues (see exercise).

Theorem

Suppose A is a square matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be eigenvectors corresponding to these eigenvalues. Then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

Proof. By induction on n.

Base case $n=1$: Trivial since there is only one vector v_1

Suppose $v_1, \cdots, v_k$ are linearly independent.

Let $\sum_{i=1}^{k+1} \alpha_i v_i = 0$. ---- $(*)$

Applying A to the both sides,

$$\sum_{i=1}^{k+1} \alpha_i \lambda_i v_i = 0 \quad \text{--- (1)}$$

Multiply λ_{k+1} to $(*)$, we have

$$\sum_{i=1}^{k+1} \alpha_i \lambda_{k+1} v_i = 0 \quad \text{--- } (2)$$

Taking (1) $-$ (2), we have.

$k+1$ th cancels out

$$\sum_{i=1}^{k} \alpha_i (\lambda_i - \lambda_{k+1}) v_i = 0$$

Since $v_1, \cdots, v_k$ are linearly independent,

$$\alpha_i \underbrace{(\lambda_i - \lambda_{k+1})}_{\neq 0} = 0 \quad \text{for } \forall\, i = 1 \sim k.$$

Since λ_i's are distinct, we must have $\alpha_i = 0$ for $\forall\, i = 1 \sim k$.

Plugging into (*), we have.

$$\alpha_{d+1} v_{d+1} = 0 \qquad \therefore \alpha_{d+1} = 0.$$

Corollary

If a $A \in M_n(\mathbb{C})$ has n distinct eigenvalues, then A is diagonalizable. That is there exists an invertible matrix $S \in M_n(\mathbb{C})$ such that $A = SDS^{-1}$, where D is a diagonal matrix with the eigenvalues of A in the diagonal.

Let v_i be eigenvector corresponding to eigenvalue λ_i,

$$S = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}, \quad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Then $AS = \begin{pmatrix} Av_1 & \cdots & Av_n \end{pmatrix} = \begin{pmatrix} \lambda_1 v_1 & \cdots & \lambda_n v_n \end{pmatrix}$
$= SD$.

Theorem

Let $A : V \to V$ be an operator with n eigenvalues. A is diagonalizable if and only if for each eigenvalue λ, the geometric multiplicity of λ and the algebraic multiplicity of λ are the same.

Example: a diagonalizable matrix

$$\begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix} \text{ is diagonalizable.}$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 8 & 1-\lambda \end{vmatrix} = (\lambda^2 - 2\lambda + 1) - 16 = \lambda^2 - 2\lambda - 15$$

$$= (\lambda - 5)(\lambda + 3) = 0 \quad \therefore \underbrace{\lambda = 5, -3}_{\text{distinct eigenvalues.}}$$

Example continued

$\lambda = -3$ $\begin{pmatrix} 4 & 2 \\ 8 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$ $2x + y = 0$

we can choose $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$,

$\lambda = 5$ $\begin{pmatrix} -4 & 2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$ $2x - y = 0$

we can choose $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Statistical Sciences
UNIVERSITY OF TORONTO

Example continued

Thus we may choose $S = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}$,

$$S^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}$$

$$S^{-1} A S = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -6 & 3 \\ 10 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -12 & 0 \\ 0 & 20 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix}.$$

Example: a matrix that is not diagonalizable

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is *not* diagonalizable.

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$$

$\lambda = 1$ with algebraic multiplicity 2.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix} = 0$$

not same.

$\ker(A - I) = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$ geometric multiplicity $= 1$

References

Howard Anton and Chris Rorres. Elementary Linear Algebra. 11th ed. Wiley, 2014

Axler S. *Linear Algebra Done Right*. 3rd ed. Undergraduate Texts in Mathematics. Springer, 2015. Available from:
https://link.springer.com/book/10.1007/978-3-319-11080-6

Treil S. *Linear Algebra Done Wrong*. 2017. Available from:
https://www.math.brown.edu/streil/papers/LADW/LADW.html