Exercises for Module 10: Differentiation and Integration

1. Show that

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

is smooth.

2. Let $f \in \mathcal{R}([a, b])$ and suppose $|f| \leq M$ for some $M > 0$. Show that $|\int_a^b f(x)dx| \leq M(b-a)$.

3. Prove the Higher-Order Leibniz product rule, i.e. for $f, g \in C^r([a,b])$ we have

$$(fg)^{(r)}(x) = \sum_{k=0}^{r} \binom{r}{k} f^{(k)}(x) g^{(r-k)}(x).$$

You can use properties of the binomial coefficient.

4. (Challenge Problem) Consider the space of continuous functions on the unit interval, $C([0,1])$. Prove that there exists a unique $f \in C([0,1])$ such that for all $x \in [0,1]$

$$f(x) = x + \int_0^x s f(s) \mathrm{d}s.$$

Hint: You can use that $C([0,1])$ is a complete metric space with respect to the supremums metric $d_\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$ for $f, g \in C([0,1])$.