Exercises for Module 3: Set Theory II and Metric Spaces I

1. Show that for sets $A, B \subseteq X$ and $f : X \to Y$, $f(A \cap B) \subseteq f(A) \cap f(B)$.

Let $A, B \subseteq X$, $f: X \to Y$. Let $y \in f(A \cap B)$. Then by definition, $\exists x \in A \cap B$ s.t. $f(x) = y$. Since $x \in A$, this means $y \in f(A)$, and since $x \in B$, this means $y \in f(B)$.
$\therefore y \in f(A) \cap f(B)$.

2. Let $f : X \to Y$ and $B \subseteq Y$. Prove that $f(f^{-1}(B)) \subseteq B$, with equality iff f is surjective.

First we show $f(f^{-1}(B)) \subseteq B$ for any $f: X \to Y$.
Let $y \in f(f^{-1}(B))$. Then $\exists x \in f^{-1}(B)$ s.t. $y = f(x)$.
Since $x \in f^{-1}(B)$, $f(x) \in B$. Thus $y = f(x) \in B$.

Next, suppose that f is surjective. We show that $B \subseteq f(f^{-1}(B))$.
Let $y \in B$. Since f is surjective, $\exists x \in X$ s.t. $f(x) = y$. Since $y \in B$, $x \in f^{-1}(B)$. Thus $y \in f(f^{-1}(B))$.

Finally we show that $f(f^{-1}(B)) = B \Rightarrow f$ is surjective.
We show the contrapositive. Suppose f is not surjective.
Then $\exists y \in Y$ s.t $f(x) \neq y$ $\forall x \in X$. However, since $f^{-1}(Y) = X$ by definition, $y \notin f(f^{-1}(Y))$. So $\exists B \subseteq Y$ (Y itself) s.t. $Y \not\subseteq f(f^{-1}(Y))$.

3. Prove that $f(\cup_{i\in I} A_i) = \cup_{i\in I} f(A_i)$, where $f : X \to Y$, $A_i \subseteq X\, \forall i \in I$.

Let $y \in f(\bigcup_{i\in I} A_i)$

$\iff \exists x \in \bigcup_{i\in I} A_i$ s.t. $f(x) = y$

$\iff \exists i^* \in I$ s.t. $x \in A_{i^*}$, $f(x) = y$

$\iff y \in f(A_{i^*})$ for some $i^* \in I$

$\iff y \in \bigcup_{i\in I} f(A_i)$

4. Show that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.

Proof.

Since $\mathbb{N} \subseteq \mathbb{Z}$, clearly we can find an injection from $\mathbb{N}$ to $\mathbb{Z}$. In particular, let $f : \mathbb{N} \to \mathbb{Z}$ be defined as $f(n) = n$. f is an injection.

It remains to show that there is an injection from $\mathbb{Z}$ to $\mathbb{N}$. Define the following function: $g : \mathbb{Z} \to \mathbb{N}$

$$g(0) = 1$$

$$\text{for } z \neq 0,\quad g(z) = \begin{cases} 2z+1 & \text{if } z > 0 \\ -2z & \text{if } z < 0 \end{cases}$$

g is an injection.

Therefore by Cantor-Bernstein, $|\mathbb{N}| = |\mathbb{Z}|$.

Note: g is in fact a bijection!

5. Show that $|(0,1)| = |(1,\infty)|$.

Let $f:(0,1) \to (1,\infty)$ be defined as $f(x) = \frac{1}{x}$.
f is a bijection.

This is probably clear, but here is a proof:

Proof

Let $\frac{1}{x} = \frac{1}{y}$. Then $x = y$. $\therefore f$ is an injection

Let $y \in (1,\infty)$. Then $x = \frac{1}{y} \in (0,1)$ is such that $f(x) = y$.
$\therefore f$ is a surjection.

6. Show that the infinity norm $||x||_\infty$, $x \in \mathbb{R}^n$, is a norm.

$$\|x\|_\infty := \max_{i=1,\dots,n} |x_i|$$

We show that $\|\cdot\|_\infty$ satisfies the 3 conditions.

(i) Positive definite

Clearly $\|x\|_\infty \geq 0$ $\forall x \in \mathbb{R}^n$ since $|x_i| > 0$ $\forall x_i \in \mathbb{R}$.
Also, if $0 = \|x\|_\infty = \max_{i=1,\dots,n} |x_i|$, then $|x_i| = 0$ $\forall i = 1,\dots,n$,
so $x = \vec{0} = (0,\dots,0)$.

(ii) Homogeneity

Let $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$.

Then $\|\alpha x\|_\infty = \max_{i=1,\dots,n} |\alpha x_i| = \max_{i=1,\dots,n} |\alpha|\,|x_i|$
$= |\alpha| \max_{i=1,\dots,n} |x_i|$
$= |\alpha|\,\|x\|_\infty$

(iii) Δ inequality

Let $x, y \in \mathbb{R}^n$.

since Δ-ineq holds for abs. value

Then $\|x+y\|_\infty = \max_{i=1,\dots,n} |x_i + y_i| \leq \max_{i=1,\dots,n} |x_i| + |y_i| = \max_{i=1,\dots,n} |x_i| + \max_{i=1,\dots,n} |y_i|$
$= \|x\|_\infty + \|y\|_\infty$

7. Let (X, d) be any metric space, and define $\tilde{d} : X \times X \to \mathbb{R}$ by

$$\tilde{d}(x,y) = \frac{d(x,y)}{1 + d(x,y)}, \quad x, y \in X.$$

Show that $\tilde{d}$ is a metric on X.

Proof. Since d is a metric, it is positive definite, symmetric, and satisfies the Δ-ineq. We show these same properties hold for $\tilde{d}$.

(i) positive definite.

$\forall x, y \in X$, we have $d(x,y) \geq 0 \Rightarrow \dfrac{d(x,y)}{1+d(x,y)} \geq 0$ and $\tilde{d}(x,y) = 0 \Leftrightarrow \dfrac{d(x,y)}{1+d(x,y)} = 0 \Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y$

(ii) symmetry

Follows from symmetry of $d(x,y)$

(iii) Δ inequality

Observe that the function $f : [0,\infty) \to \mathbb{R}$ defined by $x \mapsto \frac{x}{1+x}$ is increasing

$\left(f'(x) = \dfrac{1+x-x}{(1+x)^2} = \dfrac{1}{(1+x)^2} > 0 \quad \forall x \in [0,\infty) \right)$.

Let $x, y, z \in X$. Then

$$\begin{aligned}
\tilde{d}(x,z) &= \frac{d(x,z)}{1+d(x,z)} \\
&\leq \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} \quad \text{since } f \text{ is increasing and } d(x,z) \leq d(x,y)+d(y,z) \\
&= \frac{d(x,y)}{1+d(x,y)+d(y,z)} + \frac{d(y,z)}{1+d(x,y)+d(y,z)} \\
&\leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} = \tilde{d}(x,y) + \tilde{d}(y,z)
\end{aligned}$$

8. Let X be a set and define $d \colon X \times X \to \mathbb{R}$ by $d(x,x) = 0$ and $d(x,y) = 1$ for $x \neq y \in X$. Prove that d is a metric on X. What do open balls look like for different radii $r > 0$?

Proof $\quad d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$

Clearly d is positive definite and symmetric by definition.

To show the Δ inequality, let $x, y, z \in X$.

case 1 $\quad x = y = z$

Then $d(x,z) = 0 = d(x,y) + d(y,z)$

case 2 $\quad x = y \neq z$ or $x \neq y = z$

Then $d(x,z) = 1$ and $d(x,y) + d(y,z) = 1$

case 3 $\quad x = z \neq y$

Then $d(x,z) = 0$ and $d(x,y) + d(y,z) = 2$

case 4 $\quad x \neq y \neq z$

Then $d(x,z) = 1 \leq 2 = d(x,y) + d(y,z)$.

Open balls.

If $r \in (0,1]$, then balls are just points, i.e. $B_r(x_0) = \{x_0\}$

If $r > 1$, then the ball is the whole set, i.e. $B_r(x_0) = X$.

This means that every set in X is open!