

Module 3: Set theory and metrics

Operational math bootcamp



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Outline

- More on set theory
- Cardinality of sets
- Metrics and norms

Recall

Definition (Image and pre-image)

Let $f : X \rightarrow Y$ and $A \subseteq X$ and $B \subseteq Y$.

- The *image* of f is the set $f(A) := \{f(x) : x \in A\}$.
- The *pre-image* of f is the set $f^{-1}(B) := \{x : f(x) \in B\}$.

Definition (Surjective, injective and bijective)

Let $f : X \rightarrow Y$, where X and Y are sets. Then

- f is *injective* if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$
- f is *surjective* if for every $y \in Y$, there exists an $x \in X$ such that $y = f(x)$
- f is *bijective* if it is both injective and surjective

Proposition

Let $f : X \rightarrow Y$ and $A \subseteq X$. Prove that $A \subseteq f^{-1}(f(A))$, with equality iff f is injective.

Proof.

Cardinality

Intuitively, the *cardinality* of a set A , denoted $|A|$, is the number of elements in the set. For sets with only a finite number of elements, this intuition is correct. We call a set with finitely many elements finite.

We say that the empty set has cardinality 0 and is finite.

Proposition

If X is finite set of cardinality n , then the cardinality of $\mathcal{P}(X)$ is 2^n .

Proof.

Definition

Two sets A and B have same cardinality, $|A| = |B|$, if there exists bijection $f : A \rightarrow B$.

Example

Which is bigger, \mathbb{N} or \mathbb{N}_0 ?

Cantor-Schröder-Bernstein

Definition

We say that the cardinality of a set A is less than the cardinality of a set B , denoted $|A| \leq |B|$ if there exists an injection $f : A \rightarrow B$.

Theorem (Cantor-Bernstein)

Let A, B , be sets. If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Example

$$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$$

Proof that $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$:

Definition

Let A be a set.

- ① A is *finite* if there exists an $n \in \mathbb{N}$ and a bijection $f : \{1, \dots, n\} \rightarrow A$
- ② A is *countably infinite* if there exists a bijection $f : \mathbb{N} \rightarrow A$
- ③ A is *countable* if it is finite or countably infinite
- ④ A is *uncountable* otherwise

Example

The rational numbers are countable, and in fact $|\mathbb{Q}| = |\mathbb{N}|$.

Proof. First we show $|\mathbb{N}| \leq |\mathbb{Q}^+|$.

Next, we show that $|\mathbb{Q}^+| \leq |\mathbb{N} \times \mathbb{N}|$.

We can extend this to \mathbb{Q} as follows:

Theorem

The cardinality of \mathbb{N} is smaller than that of $(0, 1)$.

Proof.

First, we show that there is an injective map from \mathbb{N} to $(0, 1)$.

Next, we show that there is no surjective map from \mathbb{N} to $(0, 1)$. We use the fact that every number $r \in (0, 1)$ has a binary expansion of the form $r = 0.\sigma_1\sigma_2\sigma_3\dots$ where $\sigma_i \in \{0, 1\}$, $i \in \mathbb{N}$. □

Proof.

Now we suppose in order to derive a contradiction that there does exist a surjective map f from \mathbb{N} to $(0, 1)$., i.e. for $n \in \mathbb{N}$ we have $f(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n) \dots$. This means we can list out the binary expansions, for example like

$$f(1) = 0.00000000 \dots$$

$$f(2) = 0.11111111 \dots$$

$$f(3) = 0.01010101 \dots$$

$$f(4) = 0.10101010 \dots$$

We will construct a number $\tilde{r} \in (0, 1)$ that is not in the image of f . □

Proof.

Define $\tilde{r} = 0.\tilde{\sigma}_1\tilde{\sigma}_2\dots$, where we define the n th entry of \tilde{r} to be the opposite of the n th entry of the n th item in our list:

$$\tilde{\sigma}_n = \begin{cases} 1 & \text{if } \sigma_n(n) = 0, \\ 0 & \text{if } \sigma_n(n) = 1. \end{cases}$$

Then \tilde{r} differs from $f(n)$ at least in the n th digit of its binary expansion for all $n \in \mathbb{N}$. Hence, $\tilde{r} \notin f(\mathbb{N})$, which is a contradiction to f being surjective. This technique is often referred to as Cantor's diagonal argument. □

Proposition

$(0,1)$ and \mathbb{R} have the same cardinality.

Proof.



We have shown that there are different sizes of infinity, as the cardinality of \mathbb{N} is infinite but still smaller than that of \mathbb{R} or $(0,1)$. In fact, we have

$$|\mathbb{N}| < |\mathbb{N}_0| < |\mathbb{Z}| < |\mathbb{Q}| < |\mathbb{R}|.$$

Because of this, there are special symbols for these two cardinalities: The cardinality of \mathbb{N} is denoted \aleph_0 , while the cardinality of \mathbb{R} is denoted \mathfrak{c} .

In fact there are many other cardinalities, as the following theorem shows:

Theorem (Cantor's theorem)

For any set A , $|A| < |\mathcal{P}(A)|$.

Metric Spaces

Definition (Metric)

A *metric* on a set X is a function $d : X \times X \rightarrow \mathbb{R}$ that satisfies:

- (a) Positive definiteness:
- (b) Symmetry:
- (c) Triangle inequality:

A set together with a metric is called a metric space.

Example (\mathbb{R}^n with the Euclidean distance)

Definition (Norm)

A *norm* on an \mathbb{F} -vector space E is a function $\| \cdot \| : E \rightarrow \mathbb{R}$ that satisfies:

- (a) Positive definiteness:
- (b) Homogeneity:
- (c) Triangle inequality:

A vector space with a norm is called a *normed space*. A normed space is a metric space using the metric $d(x, y) = \|x - y\|$.

Example (p -norm on \mathbb{R}^n)

The p -norm is defined for $p \geq 1$ for a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ as

The infinity norm is the limit of the p -norm as $p \rightarrow \infty$, defined as

Example (p -norm on $C([0, 1]; \mathbb{R})$)

If we look at the space of continuous functions $C([0, 1]; \mathbb{R})$, the p -norm is

and the ∞ -norm (or sup norm) is

Definition

A subset A of a metric space (X, d) is *bounded* if there exists $M > 0$ such that $d(x, y) < M$ for all $x, y \in A$.

Definition

Let (X, d) be a metric space. We define the *open ball* centred at a point $x_0 \in X$ of radius $r > 0$ as

$$B_r(x_0) := \{x \in X : d(x, x_0) < r\}.$$

Example

In \mathbb{R} with the usual norm (absolute value), open balls are symmetric open intervals, i.e.

Example: Open ball in \mathbb{R}^2 with different metrics

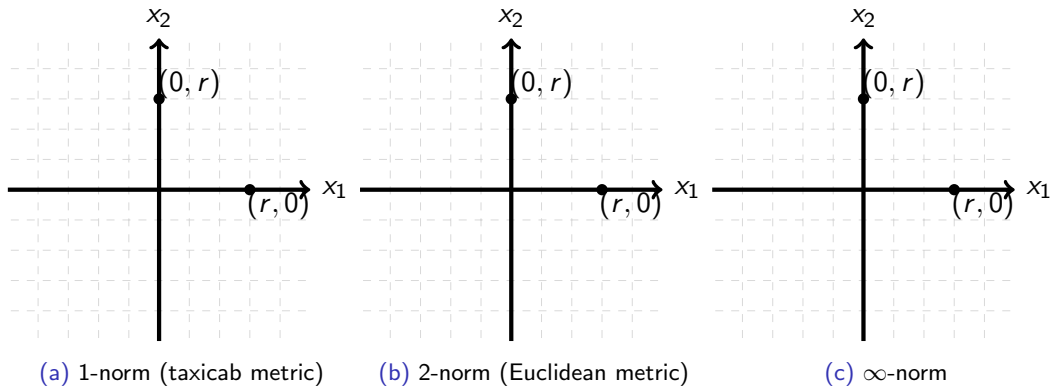


Figure: $B_r(0)$ for different metrics

References

Runde, Volker (2005). *A Taste of Topology*. Universitext. url:
<https://link.springer.com/book/10.1007/0-387-28387-0>

Zwiernik, Piotr (2022). *Lecture notes in Mathematics for Economics and Statistics*.
url: <http://84.89.132.1/piotr/docs/RealAnalysisNotes.pdf>