Module 3: Set theory and metrics

Operational math bootcamp

Ichiro Hashimoto

University of Toronto

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Outline

- More on set theory
- Cardinality of sets
- Metrics and norms

Recall

Image $A \xrightarrow{f} f(A)$

Definition (Image and pre-image)

Let $f : X \to Y$ and $A \subseteq X$ and $B \subseteq Y$.

- The *image* of f is the set $f(A) := \{f(x) : x \in A\}$.
- The *pre-image* of f is the set $f^{-1}(B) := \{x : f(x) \in B\}$.

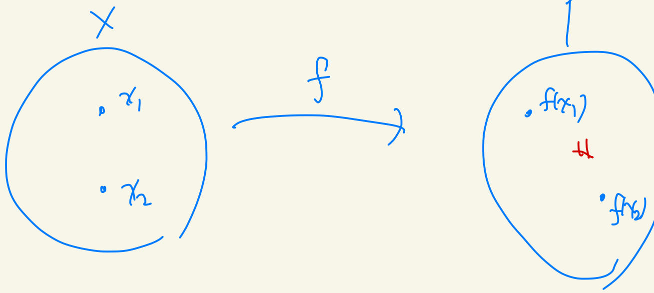
Pre-image. $f^{-1}(B) \xrightarrow{f} B$

Definition (Surjective, injective and bijective)

Let $f : X \to Y$, where X and Y are sets. Then

- f is *injective* if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$
- f is *surjective* if for every $y \in Y$, there exists an $x \in X$ such that $y = f(x)$
- f is *bijective* if it is both injective and surjective

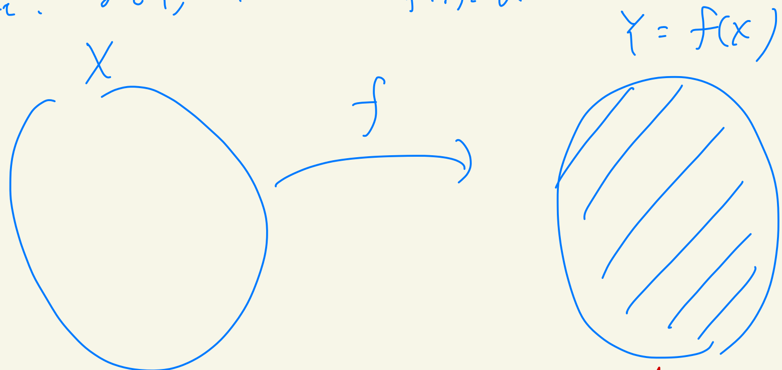
Injective : $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$



for $\forall\, x_1 \neq x_2$

Equivalently, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Surjective : $\forall\, y \in Y$, $\exists\, x \in X$ s.t. $f(x) = y$



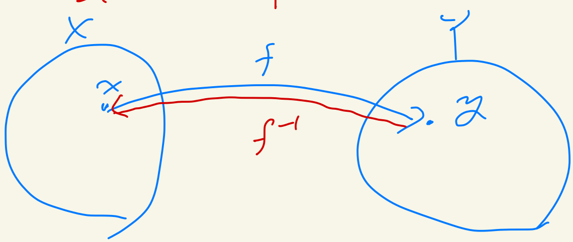
All points are filled.

In short, $Y = f(X)$

Bijective : Both injective and surjective

$\Longleftrightarrow$ $\forall\, y \in Y$, $\underbrace{\exists!}_{\text{unique}}\, x \in X$ s.t. $y = f(x)$

$\Longleftrightarrow$ Inverse map f^{-1} is well-defined



Proposition

Let $f : X \to Y$ and $A \subseteq X$. Prove that $A \subseteq f^{-1}(f(A))$, with equality ~~iff~~ if f is injective.

Proof.

We first show $A \subset f^{-1}(f(A))$.

We need to show $a \in A$ implies $\underbrace{a \in f^{-1}(f(A))}$

by definition $\iff f(a) \in f(A)$

This is true because $a \in A$.

Next we show "if" part.

We need to show $\underbrace{A \subset f^{-1}(f(A))}_{\text{already done by the first part.}}$ and $f^{-1}(f(A)) \subset A$.

Let $a \in f^{-1}(f(A))$.

This is equivalent to $f(a) \in f(A)$.

Then there exists $a' \in A$ s.t. $f(a) = f(a')$.

Since f is injective, $a = a' \in A$.

Thus, we have $f^{-1}(f(A)) \subset A$.

Cardinality

Intuitively, the *cardinality* of a set A, denoted $|A|$, is the number of elements in the set. For sets with only a finite number of elements, this intuition is correct. We call a set with finitely many elements finite.

We say that the empty set has cardinality 0 and is finite.

Proposition

If X is finite set of cardinality n, then the cardinality of $\mathcal{P}(X)$ is 2^n.

Proof. X consists of n elements

$X = \{x_1, \cdots, x_n\}$

For $A \subset X$, for each x_i → $\in A$ / → $\notin A$ } 2 options for each i.

Since $i = 1, \cdots, n$, there are 2^n distinct subsets of X.

Definition

Two sets A and B have same cardinality, $|A| = |B|$, if there exists bijection $f : A \to B$.

Example

Which is bigger, $\mathbb{N}$ or $\mathbb{N}_0$? A. $|\mathbb{N}| = |\mathbb{N}_0|$

Let $f : \mathbb{N} \to \mathbb{N}_0$ be $f(n) = n - 1$.

Then f is both injective and surjective.

therefore f is bijective and $|\mathbb{N}| = |\mathbb{N}_0|$.

Cantor-Schröder-Bernstein

If f is injective, $f: A \to f(A)$ is bijective.

Definition

We say that the cardinality of a set A is less than the cardinality of a set B, denoted $|A| \leq |B|$ if there exists an injection $f : A \to B$.

Theorem (Cantor-Bernstein)

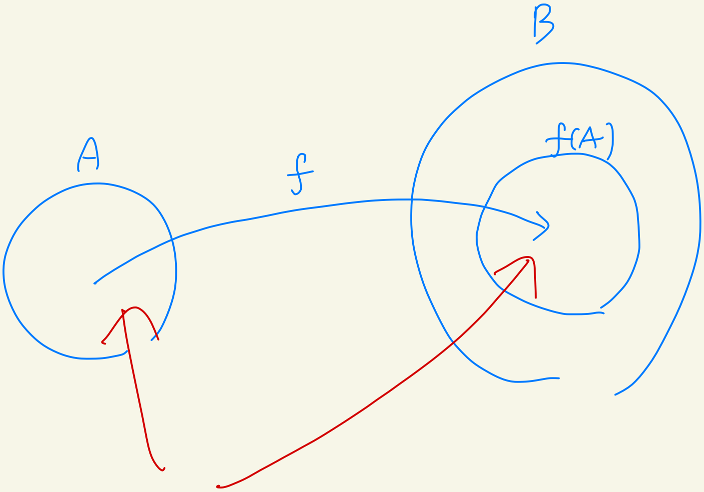
Let A, B, be sets. If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Example

$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$

this doesn't hold for finite set

If $|A| = n$, then $|A \times A| = n^2$



$f: A \to f(A)$ is bijective $\quad |A| = |f(A)|$

but $f(A) \subset B$.

So, we should define

$|A| \leq |B|$

Proof that $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$: By Cantor-Bernstein.

Let $f: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by $f(n) = (n, 1)$

Then $n \neq m$ implies $f(n) = (n, 1) \neq (m, 1) = f(m)$

Thus f is injective, which means $|\mathbb{N}| \leq |\mathbb{N} \times \mathbb{N}|$.

Let $g: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be

$g(n, m) = 2^n \cdot 3^m$

If $g(n, m) = g(n', m')$, then $2^n \cdot 3^m = 2^{n'} \cdot 3^{m'}$.

Since 2 and 3 are distinct primes, we must have $n = n'$, $m = m'$.

Thus g is injective and $|\mathbb{N} \times \mathbb{N}| \leq |\mathbb{N}|$

Definition

Let A be a set.

1. A is *finite* if there exists an $n \in \mathbb{N}$ and a bijection $f : \{1, \ldots, n\} \to A$
2. A is *countably infinite* if there exists a bijection $f : \mathbb{N} \to A$
3. A is *countable* if it is finite or countably infinite
4. A is *uncountable* otherwise

Example

The rational numbers are countable, and in fact $|\mathbb{Q}| = |\mathbb{N}|$.

Proof. First we show $|\mathbb{N}| \leq |\underline{\mathbb{Q}^+}|$.

set of positive rationals

Let $f : \mathbb{N} \to \mathbb{Q}^+$ be $f(n) = n$.

e.g. $x = \frac{2}{4} = \frac{1}{2}$

use this

Next, we show that $|\mathbb{Q}^+| \leq |\mathbb{N} \times \mathbb{N}|$.

Recall $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$.

For $\forall q \in \mathbb{Q}^+$, $\exists!\, r, s \in \mathbb{N}$ s.t. $q = \frac{r}{s}$, where r and s are mutually prime.

Let $\varphi(q) = (r, s)$

Then $\varphi : \mathbb{Q}^+ \to \mathbb{N} \times \mathbb{N}$ is injective.

Let $q_1 = r_1/s_1$, $q_2 = r_2/s_2$ and $\varphi(q_1) = \varphi(q_2)$.

Then $(r_1, s_1) = (r_2, s_2)$ $\therefore r_1 = r_2,\ s_1 = s_2 \Longrightarrow q_1 = q_2$

By Cantor-Bernstein, we know that $|\mathbb{Q}^+| = |\mathbb{N}|$

We can extend this to $\mathbb{Q}$ as follows:

Since $|\mathbb{Q}^+| = |\mathbb{N}|$, there exists $f_+ : \mathbb{Q}^+ \to \mathbb{N}$ which is bijective.

Using f_+, we can define a bijection $f_- : \mathbb{Q}^- \to \mathbb{Z}^-$ by

$$f_-(-q) = -f_+(q)$$

Then we can define bijection $f : \mathbb{Q} \to \mathbb{Z}$ by

$$f(q) = \begin{cases} f_+(q) & \text{if } q > 0 \\ 0 & \text{if } q = 0 \\ f_-(q) & \text{if } q < 0 \end{cases} \implies |\mathbb{Q}| = |\mathbb{Z}|$$

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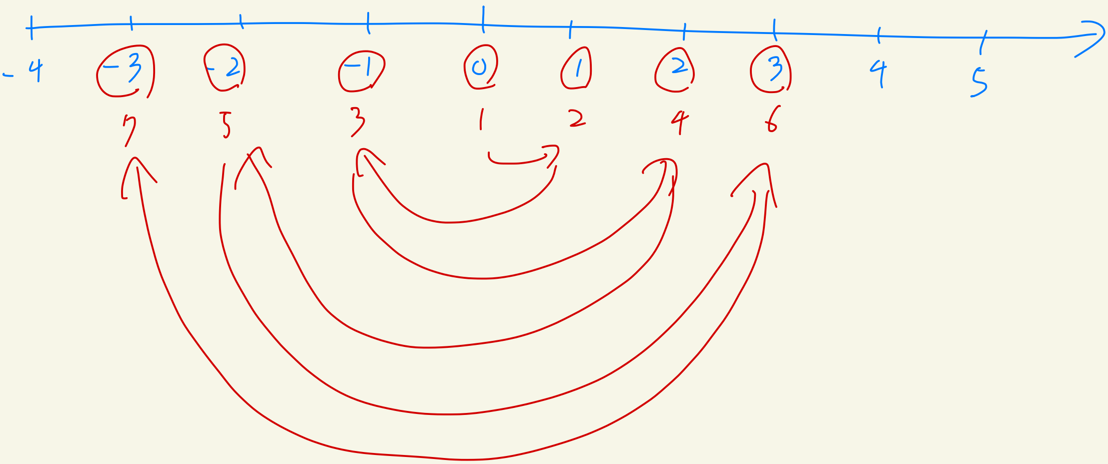
Now the proof is reduced to showing $|\mathbb{Z}| = |\mathbb{N}|$.

Let $g: \mathbb{Z} \to \mathbb{N}$ by the following way



Theorem

The cardinality of $\mathbb{N}$ is strictly smaller than that of $(0, 1)$.

Proof.

First, we show that there is an injective map from $\mathbb{N}$ to $(0, 1)$.

Let $f(n) = \frac{1}{n+1}$ $\therefore |\mathbb{N}| \leq |(0,1)|$

Next, we show that there is no surjective map from $\mathbb{N}$ to $(0, 1)$. We use the fact that every number $r \in (0, 1)$ has a binary expansion of the form $r = 0.\sigma_1\sigma_2\sigma_3\ldots$ where $\sigma_i \in \{0, 1\}$, $i \in \mathbb{N}$. □

Proof.

Now we suppose in order to derive a contradiction that there does exist a surjective map f from $\mathbb{N}$ to $(0, 1)$., i.e. for $n \in \mathbb{N}$ we have $f(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\ldots$. This means we can list out the binary expansions, for example like

$$
\begin{aligned}
f(1) &= 0.\overset{\sigma_1(1)}{0}0000000\ldots \\
f(2) &= 0.1\overset{\sigma_2(2)}{1}11111111\ldots \\
f(3) &= 0.01\overset{\sigma_3(3)}{0}1010101\ldots \\
f(4) &= 0.101\overset{\sigma_4(4)}{0}101010\ldots
\end{aligned}
$$

We will construct a number $\tilde{r} \in (0, 1)$ that is not in the image of f. $\square$

Proof.

Define $\tilde{r} = 0.\tilde{\sigma}_1\tilde{\sigma}_2\ldots$, where we define the nth entry of $\tilde{r}$ to be the the opposite of the nth entry of the nth item in our list:

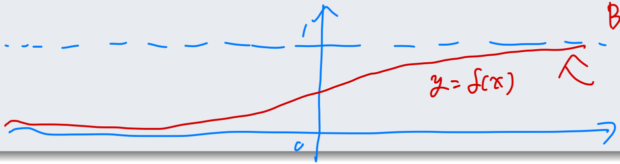
$$\tilde{\sigma}_n = \begin{cases} 1 & \text{if } \sigma_n(n) = 0, \\ 0 & \text{if } \sigma_n(n) = 1. \end{cases}$$

Then $\tilde{r}$ differs from $f(n)$ at least in the nth digit of its binary expansion for all $n \in \mathbb{N}$. Hence, $\tilde{r} \notin f(\mathbb{N})$, which is a contradiction to f being surjective. This technique is often referred to as Cantor's diagonal argument. □

Proposition

(0,1) and $\mathbb{R}$ have the same cardinality.

Proof.



Bijection $f: \mathbb{R} \to (0,1)$

monotone, continuous, $0 < f(x) < 1$

$\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to\infty} f(x) = 1$ $\square$

We have shown that there are different sizes of infinity, as the cardinality of $\mathbb{N}$ is infinite but still smaller than that of $\mathbb{R}$ or $(0,1)$. In fact, we have

$$|\mathbb{N}| = |\mathbb{N}_0| = |\mathbb{Z}| \leq |\mathbb{Q}| < |\mathbb{R}|.$$

Because of this, there are special symbols for these two cardinalities: The cardinality of $\mathbb{N}$ is denoted $\aleph_0$, while the cardinality of $\mathbb{R}$ is denoted $\mathfrak{c}$.
In fact there are many other cardinalities, as the following theorem shows:

Theorem (Cantor's theorem)

For any set A, $|A| < |\mathcal{P}(A)|$.

Metric Spaces

metric is a generalization of "distance"

Definition (Metric)

A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ that satisfies:

(a) Positive definiteness: $d(x,y) \geq 0$ for $\forall x, y$ and $d(x,y) = 0$ iff $x = y$

(b) Symmetry: $d(x,y) = d(y,x)$

(c) Triangle inequality: $d(x,y) + d(y,z) \geq d(x,z)$

A set together with a metric is called a metric space.

Example ($\mathbb{R}^n$ with the Euclidean distance)

$$d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \quad , \quad \forall x,y \in \mathbb{R}^n$$

$\mathbb{R}$ or $\mathbb{C}$

Definition (Norm)

A *norm* on an $\mathbb{F}$-vector space E is a function $\|\cdot\| : E \to \mathbb{R}$ that satisfies:

(a) Positive definiteness: $\|x\| \geq 0, \ \forall x \in E$ and $\|x\| = 0$ iff $x = 0$

(b) Homogeneity: For $\forall x \in E, \ \forall \alpha \in \mathbb{F}$ $\|\alpha x\| = |\alpha| \, \|x\|$

(c) Triangle inequality: $\forall x, y \in E$, $\|x + y\| \leq \|x\| + \|y\|$

A vector space with a norm is called a normed space. A normed space is a metric space using the metric $d(x, y) = \|x - y\|$.

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Example (p-norm on $\mathbb{R}^n$)

The p-norm is defined for $p \geq 1$ for a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ as

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

when $p = 2$, 2- norm = euclidean norm.

The infinity norm is the limit of the p-norm as $p \to \infty$, defined as

$$\|x\|_\infty = \max_i |x_i|$$

space of continuous functions

Example (p-norm on $C([0,1];\mathbb{R})$)

If we look at the space of continuous functions $C([0,1];\mathbb{R})$, the p-norm is

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p}$$

and the $\infty-$norm (or sup norm) is

$$\|f\|_\infty = \max_{x \in [0,1]} |f(x)|$$

Definition

A subset A of a metric space (X, d) is *bounded* if there exists $M > 0$ such that $d(x, y) < M$ for all $x, y \in A$.

Definition

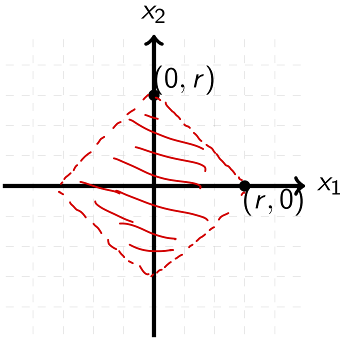
Let (X, d) be a metric space. We define the *open ball* centred at a point $x_0 \in X$ of radius $r > 0$ as

$$B_r(x_0) := \{x \in X : d(x, x_0) < r\}.$$

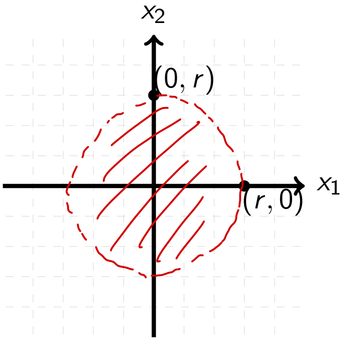
Example

In $\mathbb{R}$ with the usual norm (absolute value), open balls are symmetric open intervals, i.e.

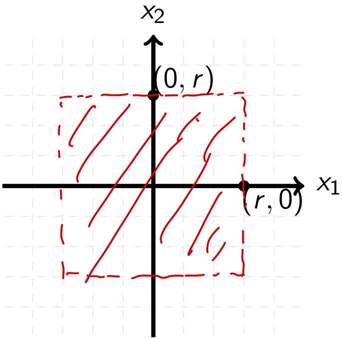
Example: Open ball in $\mathbb{R}^2$ with different metrics



(a) 1-norm (taxicab metric)



(b) 2-norm (Euclidean metric)



(c) ∞-norm

Figure: $B_r(0)$ for different metrics

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