Module 4: Metric Spaces and Sequences II

1. Show that the infinite intersection of open sets may not be open and that the infinite union of closed sets may not be closed.

2. Find the closure, interior, and boundary of the following sets using Euclidean distance:

(i) $\{(x, y) \in \mathbb{R}^2 : y < x^2\} \subseteq \mathbb{R}^2$

(ii) $[0, 1) \times [0, 1) \subseteq \mathbb{R}^2$

(iii) $\{0\} \cup \{1/n \colon n \in \mathbb{N}\} \subseteq \mathbb{R}$

3. Prove the following: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d) that converges to a point $x \in X$. Then x is unique.

4. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$ such that $x_n \to x$ and and $y_n \to y$, with $\alpha, x, y, \in \mathbb{R}$.

   (i) Show that $\alpha\, x_n \to \alpha\, x$.

   (i) Show that $x_n + y_n \to x + y$.

5. Show that discrete metric spaces (i.e. those with the metric defined as define $d\colon X \times X \to \mathbb{R}$ by $d(x,x) = 0$ and $d(x,y) = 1$ for $x \neq y \in X$) are complete.