Module 4: Metric Spaces II

Operational math bootcamp

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$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

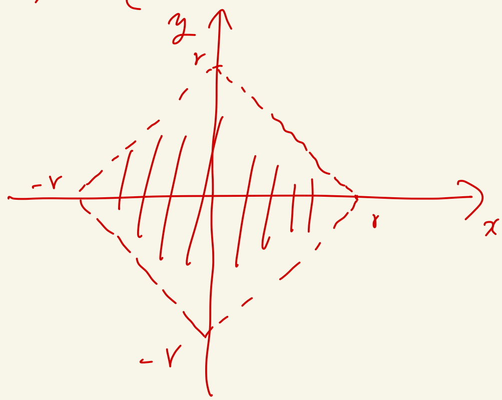
If $p=2$ $\quad \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$ usual norm

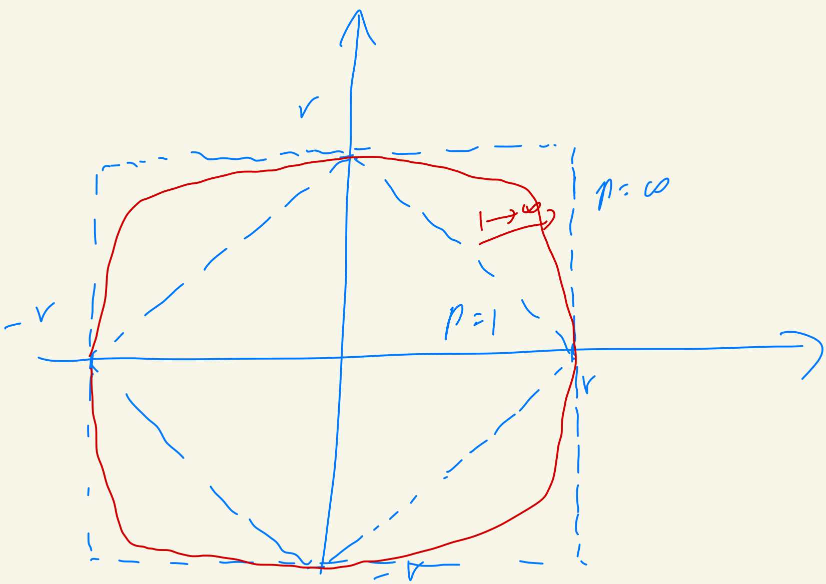
$p=1$ $\quad \|x\|_1 = \sum_{i=1}^{n} |x_i|$

$n=2$ case

$(x,y) \to \|(x,y)\|_1 = |x|+|y|$

$B_r(x,y) = \{ (x,y) \in \mathbb{R}^2 : |x|+|y| < r \}$



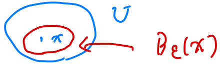


Outline

- Open and closed sets
- Sequences
  - Cauchy sequences
  - subsequences

Metric.

- $d(x,y) \geq 0$ and $d(x,y)=0$ iff $x=y$
- $d(x,y) = d(y,x)$
- $d(x,y) + d(y,z) \geq d(x,z)$



Definition (Open and closed sets)

Let (X, d) be a metric space.

- A set $U \subseteq X$ is *open* if for every $x \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$.
- A set $F \subseteq X$ is *closed* if $F^c := X \setminus F$ is open.

Note:

$B_\varepsilon(x) = \{ y \in X : d(x, y) < \varepsilon \}$

a ball in (X, d).

Proposition

Let (X, d) be a metric space.

1. *Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are open, then $A_1 \cap A_2$ is open.* $\rightarrow$ generalized to finite intersections.
2. *If $A_i \subseteq X$, $i \in I$ are open, then $\cup_{i \in I} A_i$ is open.*

can be infinite.

Proof. (1) Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are open, then $A_1 \cap A_2$ is open.

Let $x \in A_1 \cap A_2$.

Since A_1 and A_2 are open, $\exists\, \varepsilon_1, \varepsilon_2 > 0$ s.t. $B_{\varepsilon_1}(x) \subset A_1$, $B_{\varepsilon_2}(x) \subset A_2$.

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then $B_\varepsilon(x) \subset B_{\varepsilon_1}(x) \cap B_{\varepsilon_2}(x) \subset A_1 \cap A_2$.

(2) If $A_i \subseteq X$, $i \in I$ are open, then $\cup_{i\in I} A_i$ is open.

Let $x \in \bigcup_{i \in I} A_i$. $\exists\, i \in I$ s.t. $x \in A_i$.

Since A_i is open, $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset A_i \subset \bigcup_{i\in I} A_i$. //

Using DeMorgan, we immediately have the following corollary:

Corollary

Let (X, d) be a metric space.

1. *Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are closed, then $A_1 \cup A_2$ is closed.* → finite union is also closed.
2. *If $A_i \subseteq X$, $i \in I$ are closed, then $\cap_{i \in I} A_i$ is closed.*

Definition (Interior and closure)

Let $A \subseteq X$ where (X, d) is a metric space.

- The *closure* of A is $\overline{A} := \{ x \in X : \forall \varepsilon > 0,\ B_\varepsilon(x) \cap A \neq \emptyset \}$
- The *interior* of A is $\mathring{A} := \{ x \in X : \exists\ \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subset A \}$
- The *boundary* of A is $\partial A := \{ x \in X : \forall \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \cap A \neq \emptyset$ and $B_\varepsilon(x) \cap A^c \neq \emptyset \}$.

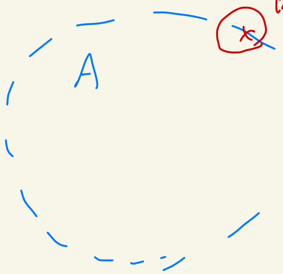
Example

Let $X = (a, b] \subseteq \mathbb{R}$ with the ordinary (Euclidean) metric. Then

$\mathring{X} = (a, b)$, $\overline{X} = [a, b]$, $\partial X = \{a, b\}$ check!

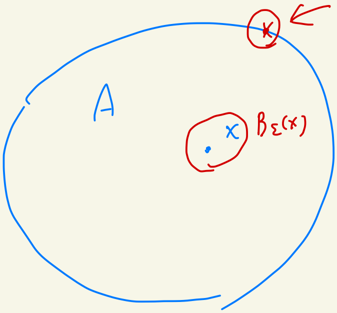
Closure

$B_\varepsilon(x)$ must intersect with A if $x \in \overline{A}$

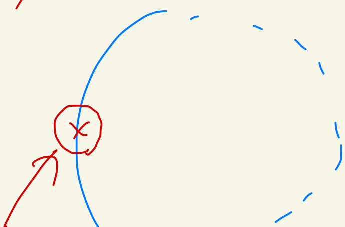


Interior

any $B_\varepsilon(x)$ intersects A^c. so $x \notin \mathring{A}$.



Boundary



$B_\varepsilon(x)$ intersects both A and A^c

need to show "$\subset$" and "$\supset$" separately.

Proposition

Let $A \subseteq X$ where (X, d) is a metric space. Then $\mathring{A} = A \setminus \partial A$.

Proof.

"$\subset$" part

Let $x \in \mathring{A}$. Then $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset A$.

contradict

Suppose $x \in \partial A$. Then by definition of ∂A, $B_\varepsilon(x) \cap A^c \neq \phi$.

This is a contradiction. $\therefore$ $\mathring{A} \subset A \setminus \partial A$.

"$\supset$" part.

Let $x \in A \setminus \partial A$. By definition of ∂A, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \cap A = \phi$ or $B_\varepsilon(x) \cap A^c = \phi$.

Since $x \in A$, we cannot have $B_\varepsilon(x) \cap A = \phi$. $\therefore$ $B_\varepsilon(x) \cap A^c = \phi$.

Thus $B_\varepsilon(x) \subset A$. $\therefore$ $x \in \mathring{A}$ //

we can say much stronger

Proposition

Let (X, d) be a metric space and $A \subseteq X$. $\overline{A}$ is closed and $\mathring{A}$ is open.

Proof. We'll prove the stronger result below.

Remark

In fact, $\underbrace{\mathring{A} = \bigcup\{U : U \text{ is open and } U \subseteq A\}}_{\text{the largest open set in } A.}$ and $\underbrace{\overline{A} = \bigcap\{F : F \text{ is closed and } A \subseteq F\}}_{\text{the smallest closed set containing } A}$.

(pf). 1) $\mathring{A} = \bigcup \{U : U \text{ open and } U \subset A\}$.

"$\subset$" part:

Let $x \in \mathring{A}$. Then, $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset A$.

Since $B_\varepsilon(x)$ is itself open, we can let $U = B_\varepsilon(x)$

to see $x \in \bigcup \{U : U \text{ open and } U \subset A\}$.

"$\supset$" part

Let $x \in \bigcup \{U : U \text{ is open and } U \subset A\}$.

Then $\exists\, U$: open s.t. $x \in U$ and $U \subset A$.

Since U is open, $\exists\, \varepsilon > 0$ s.t. $\underbrace{B_\varepsilon(x) \subset U \subset A}$.

$\Rightarrow x \in \mathring{A}$.

2) $\bar{A} = \bigcap \{F : F \text{ is closed } A \subset F\}$.

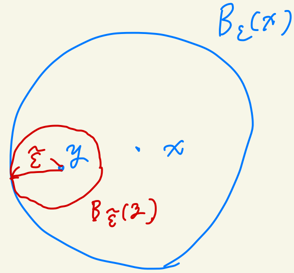
"$\subset$" part. It suffices to show $\bar{A}$ itself is closed.

Let $x \in \bar{A}^c$. Then by definition of $\bar{A}$, $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \cap A = \phi$.

That means $B_\varepsilon(x) \subset A^c$.

We need to further show $B_\varepsilon(x) \subset \bar{A}^c$.

Let $y \in B_\varepsilon(x)$ and define $\hat{\varepsilon} = \varepsilon - d(x, y)$.



Then $\underbrace{B_{\tilde{\varepsilon}}(z) \subset B_\varepsilon(x)}_{\text{by triangle inequality}} \subset A^c$.

$\therefore\ y \in \bar{A}^c$

Thus $B_\varepsilon(x) \subset \bar{A}^c$, which implies $\bar{A}$ is a closed set.

"$\supset$" part. From "$\subset$" argument, we know that $\bar{A}$ is closed.

Let F be a closed set $F \supset A$.

We must show $\bar{A} \subset F$

which is equivalent to $F^c \subset \bar{A}^c$

Let $x \in F^c \left(\subset A^c \right)$. Since F^c is open,

$\exists\ \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset F^c \subset A^c$. $\therefore B_\varepsilon(x) \cap A = \emptyset$.

By definition of $\bar{A}$, $x \notin \bar{A} \iff x \in \bar{A}^c$.

$\therefore\ F^c \subset \bar{A}^c \iff \bar{A} \subset F$.

Sequences

Definition (Sequence)

Let (X, d) be a metric space. A *sequence* is an ordered list of points x_n, $n \in \mathbb{N}$, in X, denoted $(x_n)_{n\in\mathbb{N}}$. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ *converges* to a point $x \in X$ if

$$\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } d(x_n, x) < \varepsilon, \forall n \geq n_\varepsilon.$$

Recall: $\overline{A} = \{x \in X : \forall \varepsilon > 0, \ B_\varepsilon(x) \cap A \neq \phi\}$

Proposition

Let (X, d) be a metric space, and let $A \subseteq X$. Then $\overline{A}$ is equal to the set of points in X which are limits of a sequence in A.

Proof. $\overline{A} = \{x \in X : \exists \{x_n\} \subset A \text{ s.t. } x_n \to x\}$.

" $\subset$ " part.

Let $x \in \overline{A}$. By definition, $\forall \varepsilon > 0$, $B_\varepsilon(x) \cap A \neq \phi$.

Let $\varepsilon = \frac{1}{n}$. Then $B_{\frac{1}{n}}(x) \cap A \neq \phi$.

Pick any $x_n \in B_{\frac{1}{n}}(x) \cap A$.

Then $x_n \in A$ and $d(x_n, x) < \frac{1}{n}$.

For $\forall \varepsilon > 0$, by taking $\frac{1}{n_\varepsilon} < \varepsilon \Longleftrightarrow n_\varepsilon > \varepsilon^{-1}$, then

for $\forall\ n \geq n_\varepsilon$

$$d(x_n, x) < \frac{1}{n} \leq \frac{1}{n_\varepsilon} < \varepsilon.$$

" $\supset$ " part.

Let x be the limit of $\{x_n\} \subset A$.

$\forall \varepsilon > 0$, $\exists n_\varepsilon$ s.t. $n \geq n_\varepsilon$ implies $\underline{d(x_n, x) < \varepsilon}$.

$\Longleftrightarrow x_n \in B_\varepsilon(x)$

Since $x_n \in A$, we have $x_n \in B_\varepsilon(x) \cap A$.

Thus $B_\varepsilon(x) \cap A \neq \phi$.

Therefore $x \in \overline{A}$. //

$$\bar{A} = \{x \in X : \exists \{x_n\} \subset A \text{ s.t. } x_n \to x\}.$$

$=$ is the smallest closed set containing A.

Corollary

A set $F \subseteq X$, where (X, d) is a metric space, is closed if and only if every sequence in F which converges in X converges to a point in F.

Remark: F is closed $\iff F = \bar{F}$

$\iff F = \{x \in X : \exists \{x_n\} \subset F \text{ s.t. } x_n \to x\}.$

Cluster points of a set

Definition

Let (X, d) be a metric space and $A \subseteq X$. A point $x \in X$ is a *cluster point* of A (also called accumulation point) if for every $\epsilon > 0$, $B_\epsilon(x)$ contains infinitely many points in $A \setminus \{x\}$.

x is a cluster point of A

$\Longleftarrow \quad \forall \varepsilon > 0, \quad B_\varepsilon(x) \cap [A \setminus \{x\}] \neq \phi.$

Proposition

$x \in X$ is a cluster point of $A \subseteq X$ where (X, d) is a metric space if and only if there exists a sequence of points $x_n \in \cancel{A}$ $A \setminus \{x\}$, $n \in \mathbb{N}$, such that $x_n \to x$.

Proof.

"$\Rightarrow$" part

Let x be a cluster point.

$\forall n$, let $\varepsilon_n = \frac{1}{n}$. Then $B_{\varepsilon_n}(x) \cap [A \setminus \{x\}] \neq \emptyset$.

implies we may pick $x_n \in B_{\varepsilon_n}(x) \cap [A \setminus \{x\}]$.

Then $\underbrace{d(x_n, x) < \frac{1}{n}}_{\Rightarrow\ x_n \to x}$ and $x_n \in A \setminus \{x\}$.

" $\Leftarrow$ " part

$\forall \varepsilon > 0$, $\exists n_\varepsilon$ s.t. $n \geq n_\varepsilon$ implies $d(x, x_n) < \varepsilon$.

This means $x_n \in B_\varepsilon(x)$.

At the same time $x_n \in A \setminus \{x\}$

Thus $x_n \in B_\varepsilon(x) \cap [A \setminus \{x\}]$

which means $B_\varepsilon(x) \cap [A \setminus \{x\}] \neq \emptyset$.

Therefore x is a cluster point.

Recall $\overline{A} = \{ x \in X : \exists \{x_n\} \subset A \text{ s.t. } x_n \to x \}$.

Combining the previous result with the limit characterization of closure gives the following:

Corollary

For $A \subseteq X$, (X, d) a metric space, we have

$$\overline{A} = A \cup \{x \in X : x \text{ is a cluster point of } A\}.$$

<u>Remark</u> Any isolated point of A cannot be a cluster point.

Cauchy sequences

Definition (Cauchy sequence)

Let (X, d) be a metric space. A sequence denoted $(x_n)_{n \in \mathbb{N}} \in X$ is called a *Cauchy sequence* if

$\forall \varepsilon > 0, \exists n_\varepsilon$ s.t. $n, m \geq n_\varepsilon$ implies $d(x_n, x_m) < \varepsilon$.

Convergence of x_n is not guaranteed.

Proposition

Let (X, d) be a metric space, and let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in X. Then $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

Proof. Let x is the limit of $\{x_n\}$.

$\forall \varepsilon > 0$, $\exists n_\varepsilon$ s.t. $n \geq n_\varepsilon$ implies $d(x_n, x) < \varepsilon$.

Then, $\forall n, m \geq n_\varepsilon$, by the triangle inequality,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \varepsilon + \varepsilon = 2\varepsilon.$$

Definition

A metric space where every Cauchy sequence converges (to a point in the space) is called *complete*.

Example: $\mathbb{R}$, $\mathbb{R}^n$ with usual euclidean metric are complete.

$\mathbb{Q}$. is not complete.

Proposition

Let (X, d) be a metric space, and let $Y \subseteq X$.

(i) If X is complete and if Y is closed in X, then Y is complete.

(ii) If Y is complete, then it is closed in X.

Proof. (i) Let $\{x_n\} \subset Y$ be Cauchy.

Since $\{x_n\} \subset X$ and X is complete

$\exists\, x \in X$ s.t. $x_n \to x$.

Since Y is closed every converging sequence in Y

must converge to a point in Y, i.e. $x \in Y$.

(ii) Since Y is complete, every convergent sequence in Y

(must be Cauchy).

converges to a point in Y.

This equivalent to say that Y is closed.

Subsequences

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d). Let $(n_k)_{k\in\mathbb{N}}$ be a sequence of natural numbers with $n_1 < n_2 < \cdots$. The sequence $(x_{n_k})_{k\in\mathbb{N}}$ is called a *subsequence* of $(x_n)_{n\in\mathbb{N}}$. If $(x_{n_k})_{k\in\mathbb{N}}$ converges to $x \in X$, we call x a *subsequential limit*.

Example

$((-1)^n)_{n\in\mathbb{N}}$ $\rightarrow$ $n = 2m$ subsequence. $\{1\}$

Proposition

A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, d) converges to $x \in X$ if and only if every subsequence of $(x_n)_{n\in\mathbb{N}}$ also converges to x.

Proof. $\Rightarrow$ is trivial

$\Leftarrow$: Suppose x_n does not converge to x.

$\exists\, \varepsilon > 0$, $\exists\, n_k$, $d(x_{n_k}, x) \geq \varepsilon$.

contradiction

By assumption, $x_{n_k} \to x$,

there exists k_ε s.t. $k \geq k_\varepsilon \Rightarrow d(x_{n_k}, x) < \varepsilon$.

Proof continued

References

Charles C. Pugh (2015). *Real Mathematical Analysis*. Undergraduate Texts in Mathematics. https://link-springer-com.myaccess.library.utoronto.ca/book/10.1007/978-3-319-17771-7

Runde ,Volker (2005). *A Taste of Topology*. Universitext. url: https://link.springer.com/book/10.1007/0-387-28387-0

Zwiernik, Piotr (2022). *Lecture notes in Mathematics for Economics and Statistics*. url: http://84.89.132.1/ piotr/docs/RealAnalysisNotes.pdf