Module 5: Metric spaces III

Operational math bootcamp

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Last time

Looked at open and closed sets.

Discussed sequences, which includes Cauchy sequences and subsequences:

- Convergent sequence: $x_n \to x \Leftrightarrow \forall \epsilon > 0 \; \exists \, n_\epsilon \in \mathbb{N}$ s.t. $d(x_n, x) < \epsilon$ for all $n \geq n_\epsilon$
- Cauchy sequence: $\forall \epsilon > 0 \; \exists \, n_\epsilon \in \mathbb{N}$ s.t. $d(x_n, x_m) < \epsilon$ for all $n, m \geq n_\epsilon$
- Proved that a convergent sequence is Cauchy
- Discussed complete metric spaces, where all Cauchy sequences converge (like $\mathbb{R}$ with the usual metric, absolute value)
- Proved that a sequence converges to x if and only if all subsequences converge to x

Outline for today

- Continuity
- Equivalent metrics
- Density and separability

Continuity

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$, and let $f : X \to Y$. f is *continuous* at x_0 if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X that converges to x_0, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

We say that f is continuous if it is continuous at every point in X.

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Theorem

Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$, and let $f : X \to Y$. The following are equivalent:

(i) *f is continuous at x_0*

(ii) *for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0))) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$*

(iii) *for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$*

(i) f is continuous at x_0
(ii) for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0))) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$
(iii) for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$

Proof. (i) $\Rightarrow$ (ii)

Suppose (ii) is false.

$\exists\, \varepsilon > 0$, $\forall\, \delta > 0$, $\exists\, x$ s.t. $d_X(x, x_0) < \delta$ and $d_Y(f(x), f(x_0)) \geq \varepsilon$.

Then $\forall\, n$, by taking $\delta = \frac{1}{n}$, $\exists\, x_n$ s.t. $d_X(x_n, x_0) < \frac{1}{n}$ and $d_Y(f(x_n), f(x_0)) \geq \varepsilon$.

Contradiction

Since $x_n \to x_0$, by (i) we must have $f(x_n) \to f(x_0)$.

This is a contradiction since we also have $d_Y(f(x_n), f(x_0)) \geq \varepsilon$ for $\forall\, n$.

(ii) $\Rightarrow$ (iii)

(ii) says $\forall \varepsilon>0, \exists \delta>0$ s.t. $\underbrace{x \in \{x \in X : d_X(x, x_0) < \delta\}}_{x \in B_\delta(x_0)}$ implies $\underbrace{d_Y(f(x), f(x_0)) < \varepsilon}_{f(x) \in B_\varepsilon(f(x_0))}$.

$\Longleftrightarrow$ $\forall \varepsilon>0, \exists \delta>0$ s.t. $x \in B_\delta(x_0)$ implies $\underbrace{f(x) \in B_\varepsilon(f(x_0))}_{x \in f^{-1}(B_\varepsilon(f(x_0)))}$ $\Longleftrightarrow$ $\forall \varepsilon>0, \exists \delta>0$ s.t. $B_\delta(x_0) \subset f^{-1}(B_\varepsilon(f(x_0)))$.

(iii) $\Rightarrow$ (i)

Let $x_n \to x_0$.

By (iii), $\forall \varepsilon>0$, $\exists \delta>0$ s.t. $B_\delta(x_0) \subset f^{-1}(B_\varepsilon(f(x_0)))$

Since $x_n \to x_0$, $\exists n_\delta$ s.t. $n \geq n_\delta$ implies $x_n \in B_\delta(x_0) \subset f^{-1}(B_\varepsilon(f(x_0)))$,

therefore $f(x_n) \in B_\varepsilon(f(x_0))$ $\therefore$ $d_Y(f(x_n), f(x_0)) < \varepsilon$.

Thus, we were able to show $\forall \varepsilon>0, \exists n_\varepsilon$ s.t. $n \geq n_\varepsilon$ implies

$d_Y(f(x_n), f(x_0)) < \varepsilon$.

Therefore, $f(x_n) \to f(x_0)$

Corollary

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. The following are equivalent:

(i) f is continuous

(ii) if $U \subseteq Y$ is open, then $f^{-1}(U)$ is open $\Leftarrow$ standard way to define continuity for a more general space.

(iii) if $F \subseteq Y$ is closed, then $f^{-1}(F)$ is closed

We need the following results about sets and functions:
Let X and Y be sets and $f : X \to Y$. Let $A, B \subseteq Y$. Then

1. $A \subseteq B \implies f^{-1}(A) \subseteq f^{-1}(B)$
2. $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.
(i) $\Rightarrow$ (ii): Let $U \subset Y$ be open. We want to show $f^{-1}(U)$ is open.

Let $x \in f^{-1}(U)$. Since $f(x) \in U$ and U is open,

$\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(f(x)) \subset U$. $\Rightarrow$ $f^{-1}(B_\varepsilon(f(x))) \subset f^{-1}(U)$.

By (iii) in the previous theorem, $\exists\, \delta > 0$ s.t. $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x))) \subset f^{-1}(U)$.

Since this is true for any $x \in f^{-1}(U)$, $f^{-1}(U)$ must be open. //

(ii) $\Rightarrow$ (i) $\forall x \in X$, $\forall \varepsilon > 0$, $\underline{B_\varepsilon(f(x)) \text{ is an open set}}$ in Y.

can apply (ii)

By (ii) $f^{-1}(B_\varepsilon(f(x)))$ is an open set in X.

Then, $\exists\, \delta > 0$ s.t. $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x)))$.

Since this holds for $\forall x$, f must be continuous.

(ii) $\Rightarrow$ (iii)

Let $F \subset Y$ be closed.

Since F^c is open, $f^{-1}(F^c) = X \setminus f^{-1}(F)$ is open by (ii).

Thus, $f^{-1}(F)$ must be closed.

(iii) $\Rightarrow$ (ii) Let $U \subset Y$ be open.

Since U^c is closed, $f^{-1}(U^c) = X \setminus f^{-1}(U)$ is closed by (iii).

Thus, $f^{-1}(U)$ must be open.

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.

- f is *uniformly continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that for every $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$, we have $d_Y(f(x_1), f(x_2))) < \epsilon$
- f is *Lipschitz continuous* if there exists a $K \geq 0$ such that for every $x_1, x_2 \in X$ we have $d_Y(f(x_1), f(x_2))) \leq K d_X(x_1, x_2)$

Proposition

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.

$$f \text{ is Lipschitz continuous} \Rightarrow \text{f is uniformly continuous} \Rightarrow \text{f is continuous}$$

Proof is one of your exercises.

Contraction Mapping Theorem

Definition

Let (X, d) be a metric space and let $f : X \to X$. We say that $x^* \in X$ is a *fixed point* of f if $f(x^*) = x^*$.

Definition

Let (X, d) be a metric space and let $f : X \to X$. f is a *contraction* if there exists a constant $k \in [0, 1)$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq kd(x, y)$. $\lneq d(x,y)$

Observe that a function is a contraction if and only if it is Lipschitz continuous with constant $K < 1$.

Theorem (Contraction Mapping Theorem)

Suppose that $f : X \to X$ is a contraction and the metric space X is complete. Then f has a unique fixed point x^*.

every Cauchy sequence converges

Example

Let $f : \left[-\frac{1}{3}, \frac{1}{3}\right] \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ be defined by the mapping $x \mapsto x^2$. Assume we use the standard Euclidean metric, $d(x, y) = |x - y|$. f has a unique fixed point because

$$|x^2 - y^2| = |x-y| \underbrace{|x+y|}_{\leq \frac{2}{3}} \leq \frac{2}{3} |x-y|$$

$\frac{2}{3} \in [0,1)$

Thus, we can apply Contraction mapping theorem.

Equivalent metrics

Definition (Equivalent metrics)

Two metrics d_1 and d_2 on a set X are *equivalent* if the identity maps from (X, d_1) to (X, d_2) and from (X, d_2) to (X, d_1) are continuous.

Proposition

Two metrics d_1, d_2 on a set X are equivalent if and only if they have the same open sets or the same closed sets.

U is open in (X, d_1) $\iff$ U is open in (X, d_2)

Definition

Two metrics d_1 and d_2 on a set X are *strongly equivalent* if for every $x, y \in X$, there exists constants $\alpha > 0$ and $\beta > 0$ such

$$\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y).$$

If two metrics are strongly equivalent then they are equivalent. The proof of this is one of the exercises.

d_1, d_2 are same upto constant factor

Example

We show that the Euclidean distance (induced by 2-norm) and the metric induced by the ∞-norm are equivalent on $\mathbb{R}^n$.

$$\|x-y\|_\infty = \max_i |x_i - y_i| \le \sqrt{\sum_{i=1}^n (x_i-y_i)^2} = 1 \, \|x-y\|_2$$

constant factors.

$$\|x-y\|_2 = \sqrt{\sum_{i=1}^n (x_i-y_i)^2} \le \sqrt{\sum_{i=1}^n \max_j (x_j-y_j)^2} = \sqrt{n} \cdot \|x-y\|_\infty$$

Can you think of an example that we've seen of a metric that isn't equivalent to the Euclidean metric?

Density

Definition

Let (X, d) be a metric space. A subset $A \subseteq X$ is called *dense* if $\overline{A} = X$.

Recall that $\overline{A} = \{ x \in X : \forall \varepsilon > 0, \ B_\varepsilon(x) \cap A \neq \phi \}$.

From this

$A \subset X$ is dense. $\iff \forall x \in X, \ \forall \varepsilon > 0, \ B_\varepsilon(x) \cap A \neq \phi$.

Why are dense sets important?

Examples

1. $(\mathbb{R}, |\cdot|)$

   $\mathbb{Q}$ is dense in $\mathbb{R}$

2. Let X be a set and define $d\colon X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

The only dense set in X is X itself

(exercise)

Definition

A metric space (X, d) is *separable* if it contains a countable dense subset.

Example:

$\mathbb{Q} \subset \mathbb{R}$

dense and countable

Thus $\mathbb{R}$ is a separable metric space.

Example

Define $\ell_\infty = \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{R},\ \sup_{n\in\mathbb{N}} |x_n| < \infty\}$, the space of bounded real valued sequences. Endow ℓ_∞ with a metric induced by the supremum norm, namely $d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sup_{n\in\mathbb{N}} |x_n - y_n|$. Then ℓ_∞ is **not separable** with respect to the topology induced by this metric.

Note: this proof relies on content from the cardinality section. Specifically, Cantor's theorem gives us that $|A| < |\mathcal{P}(A)|$ for any set A.

Proof. Suppose $S \subset \ell_\infty$ is a countable dense subset.

Let $S = \{ (x_n^k)_{n\in\mathbb{N}} : k \in \mathbb{N} \}$.

$k=1$: x_1^1 x_2^1 x_3^1 x_4^1 - - - - -

$k=2$: x_1^2 x_2^2 x_3^2 x_4^2 - - - - -

$k=3$: x_1^3 x_2^3 x_3^3 x_4^3 - - - - -

$k=4$: x_1^4 x_2^4 x_3^4 x_4^4 - - - -

Proof continued.

Define a new sequence $(y_n)_{n\in\mathbb{N}}$ by

$$y_n = \begin{cases} 0 & \text{if } |x_n^n| > 1 \\ x_n^n + 1 & \text{if } |x_n^n| \leq 1 \end{cases}$$

By definition $\forall n$, $|y_n| \leq 2$. Thus $(y_n)_{n\in\mathbb{N}} \in l_\infty$.

Furthermore, definition of y_n implies

$$d\left((y_n)_{n\in\mathbb{N}}, (x_n^k)_{n\in\mathbb{N}} \right) \geq 1$$

This holds for any $(x_n^k)_{n\in\mathbb{N}} \in S$.

By taking $0 < \varepsilon < 1$, then $B_\varepsilon((y_n)_{n\in\mathbb{N}}) \cap S = \phi$

Since S is dense, this is a contradiction.

(proof ②)

For any $I \subset \mathbb{N}$, define $e^I = (e^I_n)_{n\in\mathbb{N}} \in l_\infty$ by

$$e_n^I = \begin{cases} 1 & \text{if } n \in I \\ 0 & \text{if } n \notin I. \end{cases}$$

Note that if $I \neq J$, $\exists n$ s.t. $|e_n^I - e_n^J| = 1$.

Therefore, $d(e^I, e^J) = 1$ if $I \neq J$.

Thus, if $\varepsilon \in (0, \frac{1}{2})$, $B_\varepsilon(e^I) \cap B_\varepsilon(e^J) = \phi$ if $I \neq J$.

So, all $B_\varepsilon(e^I)$ are disjoint.

Let $S \subset l_\infty$ be a dense subset.

Then $B_\varepsilon(e^I) \cap S \neq \phi$ by definition

But since all $B_\varepsilon(e^I)$ are disjoint

$$|S| \geq |\{\{I : I \subset \mathbb{N}\}\}|$$ ← the number of $B_\varepsilon(e^I)$

$$= |\underbrace{\mathcal{P}(\mathbb{N})}_{\text{the power set of } \mathbb{N}}| > |\mathbb{N}|$$

Therefore S cannot be countable.

References

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