

# Module 6: Metric Spaces IV

## Operational math bootcamp



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# Outline

- Compactness
- Extra properties of  $\mathbb{R}$ 
  - Right- and left-continuity
  - Lim sup and lim inf

# Last time

## Definition

Let  $(X, d)$  be a metric space. A subset  $A \subseteq X$  is called *dense* if  $\overline{A} = X$ .

## Definition

A metric space  $(X, d)$  is *separable* if it contains a countable dense subset.

## Example

$\mathbb{R}$  is separable because  $\mathbb{Q}$  is dense in  $\mathbb{R}$

## Example

Define  $\ell_\infty = \{(x_n)_{n \in \mathbb{N}} : x_n \in \mathbb{R}, \sup_{n \in \mathbb{N}} |x_n| < \infty\}$ , the space of bounded real valued sequences. Endow  $\ell_\infty$  with a metric induced by the supremum norm, namely  $d((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) = \sup_{n \in \mathbb{N}} |x_n - y_n|$ . Then  $\ell_\infty$  is **not separable** with respect to the topology induced by this metric.

Note: this proof relies on content from the cardinality section. Specifically, Cantor's theorem gives us that  $|A| < |\mathcal{P}(A)|$  for any set  $A$ .

*Proof.*

*Proof continued.*

# Compactness

## Definition

Let  $(X, d)$  be a metric space and  $K \subseteq X$ .

A collection  $\{U_i\}_{i \in I}$  of open sets is called *open cover* of  $K$  if  $K \subseteq \bigcup_{i \in I} U_i$ .

The set  $K$  is called *compact* if for all open covers  $\{U_i\}_{i \in I}$  there exists a finite subcover, meaning there exists an  $n \in \mathbb{N}$  and  $\{U_1, \dots, U_n\} \subseteq \{U_i\}_{i \in I}$  such that  $K \subseteq \bigcup_{i=1}^n U_i$ .

## Example

Let  $S \subseteq X$  where  $(X, d)$  is a metric space. If  $S$  is finite, then it is compact.

*Proof.*

## Example

$(0, 1)$  is not compact.

*Proof.*



## Proposition

Let  $(X, d)$  be a metric space and take a non-empty subset  $K \subseteq X$ . The following holds:

- ① If  $X$  is compact and  $K$  is closed, then  $K$  is compact (i.e. closed subsets of compact sets are compact).
- ② If  $K$  is compact, then  $K$  is closed.

*Proof.* (1) If  $X$  is compact and  $K \subseteq X$  is closed, then  $K$  is compact

(2)  $K \subseteq X$  compact  $\Rightarrow K$  is closed.



Arbitrary compact metric spaces have some nice properties in general as the next proposition shows.

### Proposition

A compact metric space  $(X, d)$  is complete and separable.

Also, just as we had a sequential characterization of the closure of a set in metric spaces, we similarly have a sequential characterization of compactness.

### Theorem

*Let  $(X, d)$  be a metric space. Then  $K \subseteq X$  is compact with respect to the metric induced by  $d$  if and only if every sequence in  $K$  admits a subsequence converging to some point in  $K$ .*

# Compactness on $\mathbb{R}^n$

## Theorem (Heine-Borel Theorem)

*Let  $K \subseteq \mathbb{R}^n$ . Then  $K$  is compact with respect to the topology induced by the Euclidean distance if and only if it is closed and bounded.*

A corollary of the last two theorems is the Bolzano-Weierstrass theorem.

## Corollary (Bolzano-Weierstrass)

*Any bounded sequence in  $\mathbb{R}^n$  has a convergent subsequence.*

## Proposition

Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Suppose  $K \subseteq X$  is compact and let  $f: K \rightarrow Y$  be continuous. Then  $f(K)$  is compact.

Note: this is a generalization of the Extreme Value Theorem to metric spaces.

Recall from the set theory section:

If  $f: X \rightarrow Y$ :

- 1  $A \subseteq B \subseteq Y \Rightarrow f^{-1}(A) \subseteq f^{-1}(B)$  and  $A \subseteq B \subseteq X \Rightarrow f(A) \subseteq f(B)$
- 2  $f^{-1}(\cup_{i \in I} A_i) = \cup_{i \in I} f^{-1}(A_i)$ , where  $A_i \subseteq Y \forall i \in I$
- 3  $f(\cup_{i \in I} A_i) = \cup_{i \in I} f(A_i)$ , where  $A_i \subseteq X \forall i \in I$
- 4  $A \subseteq X \Rightarrow A \subseteq f^{-1}(f(A))$
- 5  $B \subseteq Y \Rightarrow f(f^{-1}(B)) \subseteq B$

*Proof.*





## Extra properties of $\mathbb{R}$

# Right and left continuous

Recall:  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous at  $x_0 \in \mathbb{R}$  if for all  $\epsilon > 0$  there exists a  $\delta > 0$  such that  $|x_0 - y| < \delta$  implies  $|f(x_0) - f(y)| < \epsilon$ .

## Definition

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$ .

- $f$  is *left continuous* at  $x_0 \in \mathbb{R}$  if for all  $\epsilon > 0$  there exists a  $\delta > 0$ , such  $|f(x_0) - f(x)| < \epsilon$  whenever  $x_0 - \delta < x < x_0$ .
- $f$  is *right continuous* at  $x_0 \in \mathbb{R}$  if for all  $\epsilon > 0$  there exists a  $\delta > 0$ , such  $|f(x_0) - f(x)| < \epsilon$  whenever  $x_0 < x < x_0 + \delta$ .

We say that  $f$  is left continuous if it is left continuous at all points in the domain, and similar for right continuous.

## Proposition

A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous if and only if it is left and right continuous.

*Proof.*



# Bounded sequences and monotone convergence

## Definition

Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ . We call  $(x_n)_{n \in \mathbb{N}}$  *bounded* if there exists an  $M > 0$  such that  $|x_n| < M$  for all  $n \in \mathbb{N}$ .

## Theorem (Monotone convergence theorem)

- (i) Suppose  $(x_n)_{n \in \mathbb{N}}$  is an increasing sequence, i.e.  $x_n \leq x_{n+1}$  for all  $n \in \mathbb{N}$ , and that it is bounded (above). Then the sequence converges. Furthermore,  $\lim_{n \rightarrow \infty} x_n = \sup_{n \in \mathbb{N}} x_n$ , where  $\sup_{n \in \mathbb{N}} x_n := \sup\{x_n : n \in \mathbb{N}\}$ .
- (ii) Suppose  $(x_n)_{n \in \mathbb{N}}$  is a decreasing sequence, i.e.  $x_n \geq x_{n+1}$  for all  $n \in \mathbb{N}$ , which is bounded (below). Then the sequence converges and  $\lim_{n \rightarrow \infty} x_n = \inf_{n \in \mathbb{N}} x_n := \inf\{x_n : n \in \mathbb{N}\}$ .

Convention:  $\sup A = \infty$  if  $A \subseteq \mathbb{R}$  is not bounded above and  $\inf A = -\infty$  if  $A$  is not bounded below.

### Lemma

*If  $A \subseteq B \subseteq \mathbb{R}$  is non-empty, then  $\inf A \leq \sup A$ ,  $\sup A \leq \sup B$ , and  $\inf A \geq \inf B$ .*

The proof of this follows from the definition of greatest lower and least upper bound.

## Definition

Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ . We define the *limit superior* of  $(x_n)_{n \in \mathbb{N}}$  as

$$\limsup_{n \rightarrow \infty} x_n := \lim_{n \rightarrow \infty} \sup_{k \geq n} x_k.$$

Similarly we define the *limit inferior* of  $(x_n)_{n \in \mathbb{N}}$  as

$$\liminf_{n \rightarrow \infty} x_n := \lim_{n \rightarrow \infty} \inf_{k \geq n} x_k.$$

If the sequence  $(x_n)_{n \in \mathbb{N}}$  is not bounded above, then  $\limsup_{n \rightarrow \infty} x_n = \infty$ . Similarly, if the sequence  $(x_n)_{n \in \mathbb{N}}$  is not bounded below, then  $\liminf_{n \rightarrow \infty} x_n = -\infty$ .



## Proposition

Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ .

- The sequence of suprema,  $s_n = \sup_{k \geq n} x_k$ , is decreasing and the sequence of infima,  $i_n = \inf_{k \geq n} x_k$ , is increasing.
- The limit superior and the limit inferior of a bounded sequence always exist and are finite.

*Proof.*

## Theorem

*Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ . Then the sequence converges to  $x \in \mathbb{R}$  if and only if  $\limsup_{n \rightarrow \infty} x_n = x = \liminf_{n \rightarrow \infty} x_n$ .*

Proof in notes.

We can extend this easily to a sequence of functions  $f_n: X \rightarrow \mathbb{R}$  as follows:

Define  $f = \limsup_{n \rightarrow \infty} f_n$  to be the function defined pointwise by  $f(x) = \limsup_{n \rightarrow \infty} (f_n(x))$  and similar for the limit inferior.

There also exists a set theoretic version in terms of unions and intersections which you will encounter in probability.

# References

Runde, Volker (2005). *A Taste of Topology*. Universitext. url:  
<https://link.springer.com/book/10.1007/0-387-28387-0>

