Module 6: Metric Spaces IV

Operational math bootcamp

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Outline

- Compactness
- Extra properties of $\mathbb{R}$
  - Right- and left-continuity
  - Lim sup and lim inf

Last time

Definition

Let (X, d) be a metric space. A subset $A \subseteq X$ is called *dense* if $\overline{A} = X$.

Definition

A metric space (X, d) is *separable* if it contains a countable dense subset.

Example

$\mathbb{R}$ is separable because $\mathbb{Q}$ is dense in $\mathbb{R}$

$\mathbb{Q}^n \underset{\text{dense}}{\subset} \mathbb{R}^n$ $\quad$ $\mathbb{R}^n$ is separable.

Example

Define $\ell_\infty = \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{R},\ \sup_{n\in\mathbb{N}} |x_n| < \infty\}$, the space of bounded real valued sequences. Endow ℓ_∞ with a metric induced by the supremum norm, namely $d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sup_{n\in\mathbb{N}} |x_n - y_n|$. Then ℓ_∞ is **not separable** with respect to the topology induced by this metric.

Note: this proof relies on content from the cardinality section. Specifically, Cantor's theorem gives us that $|A| < |\mathcal{P}(A)|$ for any set A.

Proof.

Proof continued.

Compactness

Definition

Let (X, d) be a metric space and $K \subseteq X$.

A collection $\{U_i\}_{i \in I}$ of open sets is called *open cover* of K if $K \subseteq \cup_{i \in I} U_i$.

The set K is called *compact* if for all open covers $\{U_i\}_{i \in I}$ there exists a finite subcover, meaning there exists an $n \in \mathbb{N}$ and $\{U_1, \ldots, U_n\} \subseteq \{U_i\}_{i \in I}$ such that $K \subseteq \cup_{i=1}^{n} U_i$.

Example

Let $S \subseteq X$ where (X, d) is a metric space. If S is finite, then it is compact.

Proof. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open covering of $S = \{x_1, \cdots, x_n\}$.

Then $S \subset \bigcup_{\lambda \in \Lambda} U_\lambda$

For each i, $\exists\, \lambda_i \in \Lambda$ s.t. $x_i \in U_{\lambda_i}$.

Then, $\underbrace{S = \bigcup_{i=1}^{n} \{x_i\} \subset \bigcup_{i=1}^{n} U_{\lambda_i}}_{}$

S is covered by a finite sub cover.

Therefore, S is compact.

Example

(0, 1) is not compact. → You need to find an open covering whose sub covers cannot cover (0,1)

Proof.

Let $U_n = \left(\frac{1}{n}, 1-\frac{1}{n}\right)$, $n \geq 2$.

Then, $\bigcup_{n\geq 2}^{\infty} U_n = (0,1)$. → $\{U_n\}_{n\geq 2}$ is an open covering of $(0,1)$.

Suppose $(0,1)$ is compact. Then, there is a finite subcover of $(0,1)$

$\{U_{n_1}, U_{n_2}, \dots, U_{n_k}\}$, $n_1 < n_2 < \dots < n_k$.

Then, $(0,1) \subset \bigcup_{j\geq 1}^{k} U_{n_j} = U_{n_k} = \left(\frac{1}{n_k}, 1-\frac{1}{n_k}\right) \subsetneq (0,1)$ contradiction!

Proposition

Let (X, d) be a metric space and take a non-empty subset $K \subseteq X$. The following holds:

1. If X is compact and K is closed, then K is compact (i.e. closed subsets of compact sets are compact).
2. If K is compact, then K is closed. $\longrightarrow$ $(0, 1)$ is not compact.

Proof. (1) If X is compact and $K \subseteq X$ is closed, then K is compact

Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open covering of K, i.e. $K \subset \bigcup_{\lambda \in \Lambda} U_\lambda$.

Since K is closed, K^c is open.

Then, $\{U_\lambda\}_{\lambda \in \Lambda} \cup \{K^c\}$ is an open cover of X.

Since X is compact, there exists a finite sub cover of X.

1) If the finite sub cover does not include K^c, then it must be a finite sub cover of K from $\{U_\lambda\}_{\lambda \in \Lambda}$.

2.) If K^c is included in the finite subcover, let the rest be $\{U_{\lambda_i}\}_{i=1,\dots,n}$.

Then $X = K \cup K^c \subset \bigcup_{i=1}^{n} U_{\lambda_i} \cup K^c$

Since $K \cap K^c = \phi$, we must have

$$\underline{K \subset \bigcup_{i=1}^{n} V_{x_i}}$$

K is covered by a finite subcover.

(2) $K \subseteq X$ compact $\Rightarrow$ K is closed.

Need to show $\overline{K} = K$.

Suppose $\overline{K} \setminus K \neq \phi$. Let $x \in \overline{K} \setminus K$.

Now consider $\overline{B_\varepsilon(x)}^c$, $\varepsilon > 0$.

Then it is easy to see. $\bigcup_{\varepsilon > 0} \overline{B_\varepsilon(x)}^c = X \setminus \{x\}$.

Since $x \notin K$, we have $K \subset \bigcup_{\varepsilon > 0} \overline{B_\varepsilon(x)}^c$.

Thus, $\left\{ \overline{B_\varepsilon(x)}^c \right\}_{\varepsilon > 0}$ is an open cover of K.

Since K is compact, there is a finite subcover of K.

Let them be $\left\{ \overline{B_{\varepsilon_1}(x)}^c, \overline{B_{\varepsilon_2}(x)}^c, \cdots, \overline{B_{\varepsilon_n}(x)}^c \right\}$,

$\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots > \varepsilon_n$. increasing order.

then, $K \subset \overline{B_{\varepsilon_n}(x)}^c$. $\Longleftrightarrow$ $\overline{B_{\varepsilon_n}(x)} \subset K^c$.

Therefore, $B_{\varepsilon_n}(x) \cap K = \phi$.

Since $x \in \overline{K}$, $B_{\varepsilon_n}(x) \cap K \neq \phi$.

contradiction !

Arbitrary compact metric spaces have some nice properties in general as the next proposition shows.

Proposition

A compact metric space (X, d) is complete and separable.

Also, just as we had a sequential characterization of the closure of a set in metric spaces, we similarly have a sequential characterization of compactness.

Theorem

Let (X, d) be a metric space. Then $K \subseteq X$ is compact with respect to the metric induced by d if and only if every sequence in K admits a subsequence converging to some point in K.

Compactness on $\mathbb{R}^n$

Theorem (Heine-Borel Theorem)

Let $K \subseteq \mathbb{R}^n$. Then K is compact with respect to the topology induced by the Euclidean distance if and only if it is closed and bounded.

↳ $(0,1)$ is not compact while $[0,1]$ is compact.

A corollary of the last two theorems is the Bolzano-Weierstrass theorem.

Corollary (Bolzano-Weierstrass)

Any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Let $\{x_n\}$ be a bounded sequence. Then $\exists M > 0$ s.t. $|x_n| \leq M$ for $\forall n$.

Then $\{x_n\} \subset K := \{|x| \leq M\}$. By Heine-Borel thm, K is compact.

Then there must be a subsequence of $\{x_n\}$ converging to some point in K.

Proposition

Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $K \subseteq X$ is compact and let $f : K \to Y$ be continuous. Then $f(K)$ is compact.

Note: this is a generalization of the Extreme Value Theorem to metric spaces.

$f: \underbrace{[a,b]}_{=K \text{ compact}} \to \mathbb{R}$ continuous. f has min and max.

$f(K)$ is compact $\Leftrightarrow$ $f(K)$ is closed and bounded.

$\Rightarrow$ $f(K)$ has min and max.

Recall from the set theory section:
If $f : X \to Y$:

1. $A \subseteq B \subseteq Y \Rightarrow f^{-1}(A) \subseteq f^{-1}(B)$ and $A \subseteq B \subseteq X \Rightarrow f(A) \subseteq f(B)$
2. $f^{-1}(\cup_{i \in I} A_i) = \cup_{i \in I} f^{-1}(A_i)$, where $A_i \subseteq Y \; \forall i \in I$
3. $f(\cup_{i \in I} A_i) = \cup_{i \in I} f(A_i)$, where $A_i \subseteq X \; \forall i \in I$
4. $A \subseteq X \Rightarrow A \subseteq f^{-1}(f(A))$
5. $B \subseteq Y \Rightarrow f(f^{-1}(B)) \subseteq B$

Proof. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of $f(K)$, i.e.

$$f(K) \subset \bigcup_{\lambda\in\Lambda} U_\lambda .$$

Then $$K \subset f^{-1}(f(K)) \subset f^{-1}\Big(\bigcup_{\lambda\in\Lambda} U_\lambda\Big)$$

$$= \bigcup_{\lambda\in\Lambda} \underline{f^{-1}(U_\lambda)}$$

open since f is continuous.

This means $\{f^{-1}(U_\lambda)\}_{\lambda\in\Lambda}$ is an open cover of K.

Since K is compact, there exists a finite subcover of K.

Let it be $\{f^{-1}(U_{\lambda_i})\}_{i=1,\dots,n}$.

Then, $K \subset \bigcup_{i=1}^{n} f^{-1}(U_{\lambda_i})$.

and so $f(K) \subset f\left(\bigcup_{i=1}^{n} f^{-1}(U_{\lambda_i})\right)$

$$= \bigcup_{i=1}^{n} f\left(f^{-1}(U_{\lambda_i})\right)$$

$$\subset \bigcup_{i=1}^{n} U_{\lambda_i}.$$

Thus, we showed that there exists a finite subcover of $f(K)$.

Extra properties of $\mathbb{R}$

Right and left continuous

Recall: $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|x_0 - y| < \delta$ implies $|f(x_0) - f(y)| < \epsilon$.

Definition

Let $f : \mathbb{R} \to \mathbb{R}$.

- f is *left continuous* at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$, such $|f(x_0) - f(x)| < \epsilon$ whenever $x_0 - \delta < x < x_0$.
- f is *right continuous* at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$, such $|f(x_0) - f(x)| < \epsilon$ whenever $x_0 < x < x_0 + \delta$.

We say that f is left continuous if it is left continuous at all points in the domain, and similar for right continuous.

Proposition

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if it is left and right continuous.

Proof.

exercise

Bounded sequences and monotone convergence

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. We call $(x_n)_{n\in\mathbb{N}}$ *bounded* if there exists an $M > 0$ such that $|x_n| < M$ for all $n \in \mathbb{N}$.

Theorem (Monotone convergence theorem)

(i) *Suppose $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence, i.e. $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, and that it is bounded (above). Then the sequence converges. Furthermore, $\lim_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} x_n$, where $\sup_{n\in\mathbb{N}} x_n := \sup\{x_n : n \in \mathbb{N}\}$.*

(ii) *Suppose $(x_n)_{n\in\mathbb{N}}$ is a decreasing sequence, i.e. $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$, which is bounded (below). Then the sequence converges and $\lim_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} x_n := \inf\{x_n : n \in \mathbb{N}\}$.*

Convention: $\sup A = \infty$ if $A \subseteq \mathbb{R}$ is not bounded above and $\inf A = -\infty$ if A is not bounded below.

Lemma

If $A \subseteq B \subseteq \mathbb{R}$ is non-empty, then $\inf A \leq \sup A$, $\sup A \leq \sup B$, *and* $\inf A \geq \inf B$.

The proof of this follows from the definition of greatest lower and least upper bound.

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. We define the *limit superior* of $(x_n)_{n\in\mathbb{N}}$ as

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \boxed{\sup_{k\geq n} x_k}.$$

a sequence indexed by n
decreases as n increases

Similarly we define the *limit inferior* of $(x_n)_{n\in\mathbb{N}}$ as

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} \boxed{\inf_{k\geq n} x_k}.$$

a sequence indexed by n.
increases as n increases.

If the sequence $(x_n)_{n\in\mathbb{N}}$ is not bounded above, then $\limsup_{n\to\infty} x_n = \infty$. Similarly, if the sequence $(x_n)_{n\in\mathbb{N}}$ is not bounded below, then $\liminf_{n\to\infty} x_n = -\infty$.

Proposition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$.

- The sequence of suprema, $s_n = \sup_{k\geq n} x_k$, is decreasing and the sequence of infima, $i_n = \inf_{k\geq n} x_k$, is increasing.
- The limit superior and the limit inferior of a bounded sequence always exist and are finite.

Proof.

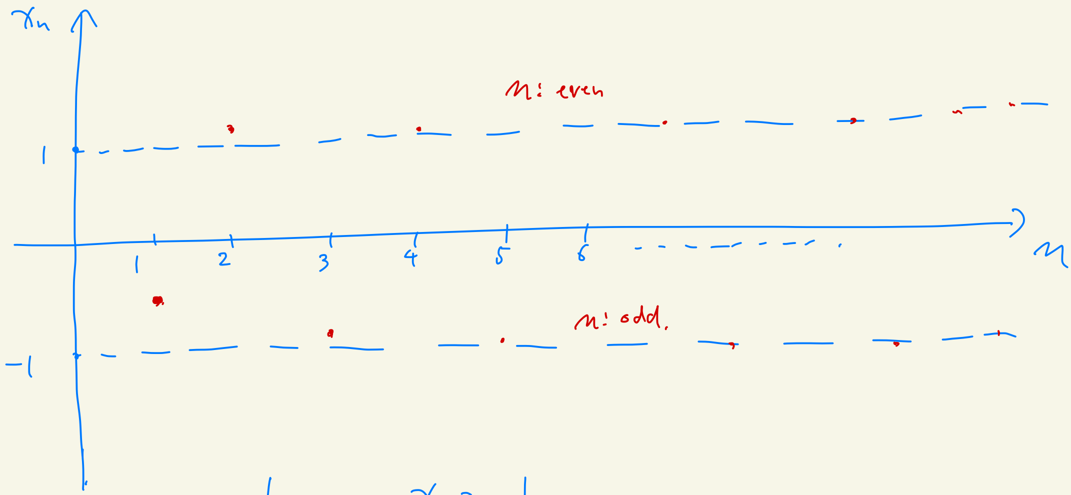
exercise

Theorem

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then the sequence converges to $x \in \mathbb{R}$ if and only if $\limsup_{n\to\infty} x_n = x = \liminf_{n\to\infty} x_n$.

Proof in notes.

Let $x_n = (-1)^n + \left(\frac{1}{2}\right)^n$



$\limsup_{n\to\infty} x_n = 1$

$\liminf_{n\to\infty} x_n = -1$

We can extend this easily to a sequence of functions $f_n \colon X \to \mathbb{R}$ as follows:

Define $f = \limsup_{n\to\infty} f_n$ to be the function defined pointwise by $f(x) = \limsup_{n\to\infty}(f_n(x))$ and similar for the limit inferior.

There also exists a set theoretic version in terms of unions and intersections which you will encounter in probability.

References

Runde, Volker (2005). *A Taste of Topology*. Universitext. url: https://link.springer.com/book/10.1007/0-387-28387-0