Exercises for Module 7: Linear Algebra I

1. Suppose that $\alpha \in \mathbb{F}, \mathbf{v} \in V$, and $\alpha\mathbf{v} = \mathbf{0}$. Prove that $a = 0$ or $\mathbf{v} = 0$.

Suppose $\alpha \neq 0$.
Since $\alpha \in \mathbb{F}$, α has a multiplicative inverse, call it α^{-1}.
Then $\alpha\vec{v} = \vec{0} \Rightarrow \alpha^{-1}\alpha\vec{v} = \vec{0} \Rightarrow \vec{v} = \vec{0}$.

Otherwise, if $\alpha = 0$, then $\alpha\vec{v} = 0\vec{v} = \vec{0}$ by lemma from class.

2. Prove the following: Let V be a vector space and let $U_1, U_2 \subseteq V$ be subspaces. Then $U_1 \cap U_2$ is also a subspace of V.

We show that the 3 properties hold.
First, since U_1, U_2 are subspaces, $\vec{0} \in U_1$ & $\vec{0} \in U_2$. Therefore $\vec{0} \in U_1 \cap U_2$.

Second, if $\vec{u}_1, \vec{u}_2 \in U_1 \cap U_2$, then $\vec{u}_1, \vec{u}_2 \in U_1$ & $\vec{u}_1, \vec{u}_2 \in U_2$. Therefore $\vec{u}_1 + \vec{u}_2 \in U_1$ and $\vec{u}_1 + \vec{u}_2 \in U_2$, since U_1, U_2 are subspaces. $\therefore \vec{u}_1 + \vec{u}_2 \in U_1 \cap U_2$.

Finally, let $\alpha \in \mathbb{F}$, $\vec{u} \in U_1 \cap U_2$. Then $\vec{u} \in U_1$ and $\alpha\vec{u} \in U_1$ and similarly $\vec{u} \in U_2$ & $\alpha\vec{u} \in U_2$. $\therefore \alpha\vec{u} \in U_1 \cap U_2$

3. Let U_1 and U_2 be subspaces of a vector space V. Prove that $U_1 \cup U_2$ is a subspace of V if and only if $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.

($\Rightarrow$) We prove the contrapositive. Suppose $U_1 \not\subseteq U_2$ and $U_2 \not\subseteq U_1$. Then we can choose $u_1 \in U_1$ s.t. s.t. $u_1 \notin U_2$ and $u_2 \in U_2$ s.t. $u_2 \notin U_1$.

Claim: $u_1 + u_2 \notin U_1$ and $u_1 + u_2 \notin U_2$

Proof of claim. Suppose $u_1 + u_2 \in U_1$. Then $u + u_2 - u_1 \in U_1$ since U_1 is a vector space. This implies $u_2 \in U_1$. Contradiction. The proof that $u_1 + u_2 \notin U_2$ is similar.

Since $u_1 + u_2 \notin U_1$ and $u_1 + u_2 \notin U_2$, $u_1 + u_2 \notin U_1 \cup U_2$. Since $u_1, u_2 \in U_1 \cup U_2$, this shows that $U_1 \cup U_2$ is <u>not</u> a subspace (not closed under addition).

($\Leftarrow$) Suppose $U_1 \subseteq U_2$. Then $U_1 \cup U_2 = U_2$, which is a subspace. Similarly, $U_2 \subseteq U_1 \Rightarrow U_1 \cup U_2 = U_1$ which is a subspace.

4. Suppose $\mathbf{v}_1, ..., \mathbf{v}_m$ is linearly independent in V and $\mathbf{w} \in V$. Prove that if $\mathbf{v}_1 + \mathbf{w}, ..., \mathbf{v}_m + \mathbf{w}$ is linearly dependent, then $\mathbf{w} \in \operatorname{span}(\mathbf{v}_1, ..., \mathbf{v}_m)$.

Proof Suppose $v_1 + w, \ldots, v_m + w$ are linearly dependent.

Then for $\alpha_i \in \mathbb{F}$, $i = 1, \ldots, m$,

$$0 = \sum_{i=1}^{m} \alpha_i (v_i + w) \quad \text{has at least one } \alpha_i \neq 0$$

$$\Rightarrow \quad 0 = \sum_{i=1}^{m} \alpha_i v_i + w \sum_{i=1}^{m} \alpha_i$$

$$\Rightarrow \quad w = \frac{\sum_{i=1}^{m} \alpha_i v_i}{\sum_{i=1}^{m} \alpha_i}$$

$$\Rightarrow w = \sum_{i=1}^{m} \frac{\alpha_i}{\sum_{j=1}^{m} \alpha_j} v_i$$

$$\Rightarrow w \in \operatorname{span}\{v_1, \ldots, v_m\}$$

5. Suppose that $\mathbf{v}_1, ..., \mathbf{v}_m$ is linearly independent in V and $\mathbf{w} \in V$. Show that $\mathbf{v}_1, ..., \mathbf{v}_m, \mathbf{w}$ is linearly independent if and only if

$$\mathbf{w} \notin \text{span}\{\mathbf{v}_1, ..., \mathbf{v}_m\}$$

($\Rightarrow$) By contrapositive.

Suppose $w \in \text{span}\{v_1, \ldots, v_m\}$. Then $\exists \alpha_i$, $i=1,\ldots,m$ s.t.

$w = \sum_{i=1}^m \alpha_i v_i$.

$\Rightarrow 0 = \sum_{i=1}^m \alpha_i v_i - w$

$\Rightarrow 0 = \sum_{i=1}^m \beta_i v_i + \beta_{m+1} w$ has a non-trivial sol'n for $\beta_i \in \mathbb{F}$

$\Rightarrow v_1, \ldots, v_m, w$ are lin. dependent

($\Leftarrow$) Also by contrapositive. Suppose $v_1, \ldots, v_m, w$ are linearly dependent. Then $\exists \alpha_i$, $i=1,\ldots,m+1$, s.t. $0 = \sum_{i=1}^m \alpha_i v_i + \alpha_{m+1} w$ has a non-trivial sol'n. Note that we must have $\alpha_{m+1} \neq 0$ because otherwise $v_1, \ldots, v_m$ would be linearly dependent. $\Rightarrow w = \sum_{i=1}^m \frac{-\alpha_i}{\alpha_{m+1}} v_i \Rightarrow w \in \text{span}\{v_1, \ldots, v_m\}$

6. Let $T \in \mathcal{L}(\mathbb{P}(\mathbb{R}), \mathbb{P}(\mathbb{R}))$ be the map $T(p(x)) = x^2 p(x)$ (multiplication by x^2).

(i) Show that T is linear.

(ii) Find the null space and range of T.

(i) Let $\alpha, \beta \in \mathbb{F}$, $p, q \in \mathbb{P}(\mathbb{R})$.

$$\begin{aligned} T(\alpha p(x) + \beta q(x)) &= x^2 (\alpha p(x) + \beta q(x)) \\ &= \alpha x^2 p(x) + \beta x^2 q(x) \\ &= \alpha T p(x) + \beta T q(x) \end{aligned}$$

(ii) Null space

We need polynomials $p(x)$ such that $x^2 p(x) = 0$ $(\forall x \in \mathbb{R})$.
This implies $p(x) = 0$ $\forall x \in \mathbb{R}$, so null $T = \{0\}$.

Range

We need to find all polynomials p s.t. $\exists$ polynomial q with
$p(x) = Tq(x) \Rightarrow p(x) = x^2 q(x)$

This holds as long as p has minimum degree ≥ 2, so
range $T = \{0\} \cup \{p(x) : \text{minimum degree of } p \text{ is at least } 2\}$.

7. Let U and V be finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Show that

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$$

<u>Proof</u> By the rank-nullity thm, for $T : U \to V$, $\dim U = \dim \operatorname{range} T + \dim \operatorname{null} T$.
Note that $T: U \to V$, $S: V \to W$, $ST: U \to W$.
Also, null ST is a subspace of U. Let T' be T restricted to the subspace null ST.

$$\begin{aligned} \dim \operatorname{null} ST &= \dim \operatorname{null} T' + \dim \operatorname{range} T' && \text{by rank nullity} \\ &= \dim \operatorname{null} T + \dim \operatorname{range} T' && \text{since null } T \subseteq \text{null } ST \\ &\leq \dim \operatorname{null} T + \dim \operatorname{null} S + \dim \operatorname{range} S(T') && \text{by rank nullity applied to range } T \\ &= \dim \operatorname{null} T + \dim \operatorname{null} S && \text{by construction} \end{aligned}$$