Exercises for Module 7: Linear Algebra I

1. Suppose that $\alpha \in \mathbb{F}, \mathbf{v} \in V$, and $\alpha\mathbf{v} = \mathbf{0}$. Prove that $a = 0$ or $\mathbf{v} = 0$.

2. Prove the following: Let V be a vector space and let $U_1, U_2 \subseteq V$ be subspaces. Then $U_1 \cap U_2$ is also a subspace of V.

3. Let U_1 and U_2 be subspaces of a vector space V. Prove that $U_1 \cup U_2$ is a subspace of V if and only if $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.

4. Suppose $\mathbf{v}_1, ..., \mathbf{v}_m$ is linearly independent in V and $\mathbf{w} \in V$. Prove that if $\mathbf{v}_1 + \mathbf{w}, ..., \mathbf{v}_m + \mathbf{w}$ is linearly dependent, then $\mathbf{w} \in \text{span}(\mathbf{v}_1, ..., \mathbf{v}_m)$.

5. Suppose that $\mathbf{v}_1, ..., \mathbf{v}_m$ is linearly independent in V and $\mathbf{w} \in V$. Show that $\mathbf{v}_1, ..., \mathbf{v}_m, \mathbf{w}$ is linearly independent if and only if

$$\mathbf{w} \notin \text{span}\{\mathbf{v}_1, ..., \mathbf{v}_m\}$$

6. Let $T \in \mathcal{L}(\mathbb{P}(\mathbb{R}), \mathbb{P}(\mathbb{R}))$ be the map $T(p(x)) = x^2 p(x)$ (multiplication by x^2).

(i) Show that T is linear.

(ii) Find the null space and range of T.

7. Let U and V be finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Show that

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$$