Exercises for Module 8: Linear Algebra II

1. Let $D \in \mathcal{L}(\mathbb{P}_4(\mathbb{R}), \mathbb{P}_3(\mathbb{R}))$ be the differentiation map, $Dp = p'$. Find bases of $\mathbb{P}_4(\mathbb{R})$ and $\mathbb{P}_3(\mathbb{R})$ such that the matrix representation of $\mathcal{M}(D)$ with respect to these basis is given by

$$\mathcal{M}(D) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Basis for $\mathbb{P}_4(\mathbb{R})$: $u_1 = \frac{1}{4}x^4$, $u_2 = \frac{1}{3}x^3$, $u_3 = \frac{1}{2}x^2$, $u_4 = x$, $u_5 = 1$

Basis for $\mathbb{P}_3(\mathbb{R})$: $v_1 = x^3$, $v_2 = x^2$, $v_3 = x$, $v_4 = 1$

Then
$$\left.\begin{aligned} T(u_1) &= \left(\tfrac{1}{4}x^4\right)' = x^3 = v_1 \\ T(u_2) &= \left(\tfrac{1}{3}x^3\right)' = x^2 = v_2 \\ T(u_3) &= \left(\tfrac{1}{2}x^2\right)' = x = v_3 \\ T(u_4) &= (x)' = 1 = v_4 \\ T(u_5) &= (1)' = 0 \end{aligned}\right\} \Rightarrow M(D) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

2. Show that matrix multiplication of square matrices is not commutative, i.e find matrices $A, B \in M_2$ such that $AB \neq BA$.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

$$AB = \begin{pmatrix} 19 & 24 \\ 43 & 50 \end{pmatrix}; \quad BA = \begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix}$$

3. A square matrix is called *nilpotent* if $\exists k \in \mathbb{N}$ such that $A^k = 0$. Show that for a nilpotent matrix A, $|A| = 0$.

Suppose A is nilpotent, i.e. $\exists k \in \mathbb{N}$ s.t $A^k = 0$.

Then $\det(A^k) = 0 \Rightarrow \det(\underbrace{A \cdots A}_{k \text{ times}}) = 0$

$\Rightarrow \underbrace{\det(A) \cdots \det(A)}_{k \text{ times}} = 0$ by properties of determinant

$\Rightarrow \det(A)^k = 0$

4. A real square matrix Q is called *orthogonal* if $Q^T Q = I$. Prove that if Q is orthogonal, then $|Q| = \pm 1$.

$$Q^T Q = I$$

$$\Rightarrow \det(Q^T Q) = \det(I)$$

$$\Rightarrow \det(Q^T)\det(Q) = 1$$

$$\Rightarrow \det(Q)\det(Q) = 1$$

$$\Rightarrow \det(Q)^2 = 1$$

$$\Rightarrow \det(Q) = \pm 1$$

5. An $n \times n$ matrix is called *antisymmetric* if $A^T = -A$. Prove that if A is antisymmetric and n is odd, then $|A| = 0$.

$$A^T = -A$$

$$\Rightarrow \det(A^T) = \det(-A)$$

$$\Rightarrow \det(A) = (-1)^n \det(A)$$

$\Rightarrow \det(A) = 0$ if n is odd

6. Let V be an inner product space, U a vector space and $S\colon U \to V$, $T\colon U \to V$ be linear maps. Show that $\langle S\mathbf{u}, \mathbf{v}\rangle = \langle T\mathbf{u}, \mathbf{v}\rangle$ for all $\mathbf{u} \in U$ and $\mathbf{v} \in V$ implies $S = T$.

Proof

Suppose $\langle Su, v\rangle = \langle Tu, v\rangle \quad \forall u \in U, v \in V$

$\Rightarrow \langle Su, v\rangle - \langle Tu, v\rangle = 0$

$\Rightarrow \langle Su - Tu, v\rangle = 0$ by linearity in 1st argument

$\Rightarrow Su - Tu = 0$ by proposition 5.57

$\Rightarrow Su = Tu \quad \forall u \in U$

$\Rightarrow S = T$